IC/2000/14

THE GENERALIZED ABEL-PLANA FORMULA. APPLICATIONS TO BESSEL FUNCTIONS AND CASIMIR EFFECT

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February 28, 2000

Abstract

One of the most efficient methods to obtain the vacuum expectation values for the physical observables in the Casimir effect is based on using the Abel-Plana summation formula. This allows to derive the regularized quantities by manifestly cutoff independent way and to present them in the form of strongly convergent integrals. However the application of Abel-Plana formula in usual form is restricted by simple geometries when the eigenmodes have a simple dependence on quantum numbers. The author generalized the Abel-Plana formula which essentially enlarges its application range. Based on this generalization, formulae have been obtained for various types of series over the zeros of some combinations of Bessel functions and for integrals involving these functions. It have been shown that these results generalize the special cases existing in literature. Further the derived summation formulae have been used to summarize series arising in the mode summation approach to the Casimir effect for spherically and cylindrically symmetric boundaries. This allows to extract the divergent parts from the vacuum expectation values for the local physical observables in the manifestly cutoff independent way. Present paper reviews these results. Some new considerations are added as well.

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1 Introduction

The Casimir effect is among the most interesting consequences of quantum field theory and is essentially the only macroscopic manifestation of the nontrivial properties of the physical vacuum. These properties may be determined from the responce of the vacuum state to classical external fields or constraints. The simplest case is realized by boundary conditions on quantized fields. Such conditions modify the zero point mode spectrum and as a result can change the energy of the vacuum. This change is manifest as an observable Casimir energy. Since the original work by Casimir in 1948 [1] many theoretical and experimental works have been done on this problem, including various types of boundary geometry and non-zero temperature effects (see, e.g., [2, 3, 4, 5] and references therein). Many different approaches have been used: mode summation method, Green function formalism, multiple scattering expansions, heat-kernel series, zeta function regularization technique, etc.. From the general theoretical point of view the main point here is the unique separation and subsequent removing of the divergences. Within the framework of the mode summation method in calculations of the expectation values for physical observables, such as energy-momentum tensor, one often needs to sum over the values of a certain function at integer points, and then subtract the corresponding quantity for unbounded space (usually presented in terms of integrals). Practically, the sum and integral, taken separately, diverge and some physically motivated procedure, to handle finite result, is needed. For a number of geometries one of the most convenient methods to obtain such regularized values of the mode sums is based on the using of the Abel-Plana formula (APF) [6, 7, 8]. In [9] this formula have been used to regularize scalar field energy momentum-tensor on backgrounds of various Friedmann cosmological models. Further applications to the Casimir effect for flat boundary geometries with corresponding references can be found in [2]. Abel-Plana formula allows (i) to extract by cutoff independent way the Minkowski vacuum part and (ii) to obtain for the regularized part strongly convergent integrals, useful, in particular, for numerical calculations. However the applications of APF in usual form is restricted by the flat boundary cases when the eigenmodes have simple dependence on quantum numbers.

In [10, 11] the APF was generalized (see also [12]). The generalized version contains two meromorphic functions. Choosing one of these functions in specific form APF in usual form is obtained. By applying the generalized formula to Bessel functions in [11, 12] summation formulae are obtained over the zeros of various combinations of these functions. In particular, formulae for Fourier-Bessel and Dini series are derived. From these formulae by specifying the constants and choosing the order of Bessel function equal to 1/2 one obtains a simple generalization of APF for the case of a function having poles. It have been shown that from generalized formula interseting results can be derived for infinite integrals involving Bessel functions. Further the obtained summation formulae are applied to regularize the vacuum expectation values for the energy-momentum tensor components of the electromagnetic field in the Casimir effect with spherically [13, 14, 15, 16] and cylindrically symmetric [17, 18] boundaries. As in the case of flat boundaries the using of generalised Abel-Plana formula allow to extract in manifestly cutoff independent way the contribution of the unbounded space and to present regularized values in terms of exponentially converging integrals.

The present paper reviews these results and is organized as follows. In section 2 the generalized Abel-Plana formula is derived and as a special case usual APF is obtained. It is indicated how to generalize this formula for the functions having poles. The applications of generalized formula to Bessel functions are considered in the next section. We derive two formulae for the sums over zeros of $AJ_\nu(z) + BzJ'_\nu(z)$. Specific examples of applications of the general formulae are considered. For $\nu = 1/2$, $B = 0$ and for analytic function $f(z)$ the APF is obtained. In section 4 from generalized Abel-Plana formula by special choice of function $g(z)$ summation formulae are derived for the series over zeros of the function $J_\nu(z)Y_\nu(\lambda z) - J_\nu(\lambda z)Y_\nu(z)$ and

similar combinations with Bessel functions derivatives. Special examples are considered. The applications to the integrals involving Bessel functions and some their combinations are discussed in section 5. A number of interesting results for these integrals are presented. Specific examples of applying these general formulae are described in the next section. In section 7 by using generalized Abel-Plana formula two theorems are proved for the integrals involving the function $J_\nu(z)Y_\mu(\lambda z) - J_\mu(\lambda z)Y_\nu(z)$ and their applications are considered. The following sections are devoted to the applications of generalized formula for the calculations of the regularized vacuum expectation values of the electromagnetic energy-momentum tensor inside (section 8) and outside (section 9) a perfectly conducting spherical shell, and for the region between two perfectly conducting spherical surfaces (section 10). In sections 11-13 the similar problems for the cylindrical surfaces are considered. The section 14 concludes the main results considered in this paper.

2 Generalized Abel-Plana formula

Let $f(z)$ and $g(z)$ be meromorphic functions for $a \leq x \leq b$ in the complex plane $z = x + iy$. Let us note by $z_{f,k}$ and $z_{g,k}$ the poles of $f(z)$ and $g(z)$ in region $a < x < b$, respectively. Assume that $\mathrm{Im} z_{f,k} \neq 0$ (see however the Remark to Lemma).

Lemma. *If functions $f(z)$ and $g(z)$ satisfy condition*

$$\lim_{h\to\infty} \int_{a\pm ih}^{b\pm ih} [g(z) \pm f(z)]\, dz = 0, \tag{2.1}$$

then the following formula takes place

$$\int_a^b f(x)dx = R[f(z), g(z)] - \frac{1}{2}\int_{-i\infty}^{+i\infty} [g(u) + \mathrm{sgn}(\mathrm{Im} z) f(u)]_{u=a+z}^{u=b+z}\, dz, \tag{2.2}$$

where

$$R[f(z), g(z)] = \pi i \left[\sum_k \mathrm{Res}_{z=z_{g,k}} g(z) + \sum_k \mathrm{Res}_{\mathrm{Im} z_{f,k}>0} f(z) - \sum_k \mathrm{Res}_{\mathrm{Im} z_{f,k}<0} f(z) \right]. \tag{2.3}$$

Proof. Let us consider a rectangle C_h with vertices $a \pm ih$, $b \pm ih$ described in the positive sense. In accordance to the residue theorem

$$\int_{C_h} g(z)dz = 2\pi i \sum_k \mathrm{Res}_{z=z_{g,k}} g(z), \tag{2.4}$$

where rhs contains sum over poles within C_h. Let C_h^+ and C_h^- denote the upper and lower halfs of this contour. Then one has

$$\int_{C_h} g(z)dz = \int_{C_h^+} [g(z) + f(z)]dz + \int_{C_h^-} [g(z) - f(z)]dz - \int_{C_h^+} f(z)dz + \int_{C_h^-} f(z)dz. \tag{2.5}$$

By the same residue theorem

$$\int_{C_h^-} f(z)dz - \int_{C_h^+} f(z)dz = 2\int_a^b f(x)dx + 2\pi i \left[\sum_k \mathrm{Res}_{\mathrm{Im} z_{f,k}<0} f(z) - \sum_k \mathrm{Res}_{\mathrm{Im} z_{f,k}>0} f(z) \right]. \tag{2.6}$$

Then

$$\int_{C_h^\pm}[g(z)\pm f(z)]dz=\pm\int_0^{\pm ih}[g(u)\pm f(u)]_{u=a+z}^{u=b+z}dz\mp\int_{a\pm ih}^{b\pm ih}[g(z)\pm f(z)]dz. \tag{2.7}$$

Combining these results and allowing in (2.4) $h\to\infty$ one obtains the formula (2.2). ■

If the functions $f(z)$ and $g(z)$ have poles with $\mathrm{Re}z_{i,k}=a,b$ $(i=f,g)$ the contour have to pass round these points on the right or left, correspondingly.

Remark. The formula (2.2) is valid also when the function $f(z)$ has real poles $z_{f,n}^{(0)}$, $\mathrm{Im}z_{f,n}^{(0)}=0$ in the region $a<\mathrm{Re}z<b$ if the main part of its Laurent expansion near of these poles does not contain even powers of $z-z_{f,n}^{(0)}$. In this case on the left of the formula (2.2) the integral is meant in the sense of the principal value, which exists as a consequence of the abovementioned condition. For brevity let us consider the case of a single pole $z=z_0$. One has

$$\int_{C_h^-}f(z)dz-\int_{C_h^+}f(z)dz=2\left[\int_a^{z_0-\rho}f(z)dz+\int_{z_0+\rho}^b f(z)dz\right]+$$
$$+2\pi i\left[\sum_k \mathrm{Res}_{\mathrm{Im}z_{f,k}<0}f(z)-\sum_k \mathrm{Res}_{\mathrm{Im}z_{f,k}>0}f(z)\right]+\int_{\Gamma_\rho^+}f(z)dz+\int_{\Gamma_\rho^-}f(z)dz, \tag{2.8}$$

with contours Γ_ρ^+ and Γ_ρ^- being the upper and lower circular arcs (with center at $z=z_0$) joining the points $z_0-\rho$ and $z_0+\rho$. By taking into account that for odd negative l

$$\int_{\Gamma_\rho^+}(z-z_0)^l dz+\int_{\Gamma_\rho^-}(z-z_0)^l dz=0, \tag{2.9}$$

in the limit $\rho\to 0$ we obtain the required result. ■

In the following on the left of (2.2) we will write p.v. $\int_a^b f(x)dx$, assuming that this integral converges in the sense of the principal value. As a direct consequence of Lemma one obtains [11]:

Theorem 1. *If in addition to the conditions of Lemma one has*

$$\lim_{b\to\infty}\int_b^{b\pm i\infty}[g(z)\pm f(z)]\,dz=0, \tag{2.10}$$

then

$$\lim_{b\to\infty}\left\{\text{p.v.}\int_a^b f(x)dx-R[f(z),g(z)]\right\}=\frac{1}{2}\int_{a-i\infty}^{a+i\infty}[g(z)+\mathrm{sgn}(\mathrm{Im}z)f(z)]\,dz, \tag{2.11}$$

where on the left $R[f(z),g(z)]$ *is defined as (2.3),* $a<\mathrm{Re}z_{f,k},\mathrm{Re}z_{g,k}<b$*, and summation goes over poles* $z_{f,k}$ *and* $z_{g,k}$ *arranged in order* $\mathrm{Re}z_{i,k}\le\mathrm{Re}z_{i,k+1}$*,* $i=f,g$*.*

Proof. To proof it is sufficient to insert in the general formula (2.2) $b\to\infty$ and to use the condition (2.10). The order of summation in $R[f(z),g(z)]$ is determined by the choice of the integration contour C_h and by limiting transition $b\to\infty$. ■

We will call the formula (2.11) as **Generalized Abel-Plana Formula** (GAPF) as for $b=n+a$, $0<a<1$, $g(z)=-if(z)\cot\pi z$ and analytic functions $f(z)$ from (2.11) follows the Abel-Plana formula (APF) [6, 7, 8]

$$\lim_{n\to\infty}\left[\sum_1^n f(s)-\int_a^{n+a}f(x)dx\right]=\frac{1}{2i}\int_a^{a-i\infty}f(z)(\cot\pi z-i)dz-\frac{1}{2i}\int_a^{a+i\infty}f(z)(\cot\pi z+i)dz. \tag{2.12}$$

The useful form of (2.12) may be obtained performing the limit $a \to 0$. By taking into account that the point $z = 0$ is a pole for integrands and therefore have to be around by arcs of the small circle C_ρ on the right and performing $\rho \to 0$ one obtains

$$\sum_{n=0}^{\infty} f(n) = \int_0^\infty f(x)dx + \frac{1}{2}f(0) + i\int_0^\infty \frac{f(ix) - f(-ix)}{e^{2\pi x} - 1}dx. \tag{2.13}$$

Note that now the condition (2.1) is satisfied if

$$\lim_{y\to\infty} e^{-2\pi|y|}|f(x+iy)| = 0 \tag{2.14}$$

uniformly in any finite interval of x. The (2.13) is the most frequently used form of APF in its physical applications. Another useful form (in particular for fermionic field calculations) to sum over the values of an analytic function at half of an odd integer points can be obtained from (2.13) [2, 19]:

$$\sum_{n=0}^{\infty} f(n+1/2) = \int_0^\infty f(x)dx - i\int_0^\infty \frac{f(ix) - f(-ix)}{e^{2\pi x} + 1}dx \tag{2.15}$$

By adding to the rhs of (2.13) the term

$$\pi i \left\{ \sum_k \mathrm{Res}_{\mathrm{Im} z_{f,k}>0} f(z) - \sum_k \mathrm{Res}_{\mathrm{Im} z_{f,k}<0} f(z) - i\sum_k \mathrm{Res}_{z=z_{f,k}} \left[f(z)\cot \pi z\right] \right\} \tag{2.16}$$

the APF may be generalized for the case when the function $f(z)$ has poles $z_{f,k}$, $\mathrm{Re} z_{f,k} > 0$, $z_{f,k} \neq 1, 2, \ldots$.

As a next consequence of (2.11) a summation formula can be obtained over the points z_n, $\mathrm{Re} z_n > 0$ at which the analytic function $s(z)$ takes integer values, $s(z_n)$ is an integer, and $s'(z_n) \neq 0$. Taking in (2.11) $g(z) = -if(z)\cot \pi s(z)$ one obtains the following formula [10]

$$\sum \frac{f(z_n)}{s'(z_n)} = w + \int_0^\infty f(x)dx + \int_0^\infty \left[\frac{f(ix)}{e^{-2\pi i s(ix)} - 1} - \frac{f(-ix)}{e^{2\pi i s(-ix)} - 1}\right] dx, \tag{2.17}$$

where

$$w = \begin{cases} 0, & \text{if} \quad s(0) \neq 0, \pm 1, \pm 2, \ldots \\ f(0)/[2s'(0)], & \text{if} \quad s(0) = 0, \pm 1, \pm 2, \ldots \end{cases} \tag{2.18}$$

For $s(z) = z$ we return to APF in usual form. An example of applications of this formula to the Casimir effect is given in [10].

3 Applications to Bessel functions

The formula (2.11) contains two meromorphic functions and is too general. To obtain more special consequences we have to specify the one of them. As we have seen in previous section the one of the possible ways leads to APF. Here we will consider another choices of the function $g(z)$ and will obtain useful formulae for the sums over zeros of Bessel function and their combinations, as well as some formulae for integrals involving these functions.

First of all to simplify the formulae let us introduce the notation

$$\bar{F}(z) \equiv AF(z) + BzF'(z) \tag{3.1}$$

for a given function $F(z)$, where the prime denotes derivative with respect to the argument of function, A and B are constants. As a function $g(z)$ in GAPF let us choose

$$g(z) = i\frac{\bar{Y}_\nu(z)}{\bar{J}_\nu(z)} f(z), \tag{3.2}$$

where $J_\nu(z)$ and $Y_\nu(z)$ are Bessel functions of the first and second (Neumann function) kind. For the sum and difference on the right of (2.11) one obtains

$$f(z) - (-1)^k g(z) = \frac{\bar{H}_\nu^{(k)}(z)}{\bar{J}_\nu(z)} f(z), \quad k = 1, 2 \tag{3.3}$$

with $H_\nu^{(1)}$ and $H_\nu^{(2)}$ being Bessel functions of the third kind or Hankel functions. For such a choice the integrals (2.1) and (2.10) can be estimated by using the asymptotic formulae for Bessel functions for fixed ν and $|z| \to \infty$ (see, for example, [20, 21]). It can be easily seen that conditions (2.1) and (2.10) are satisfied if the function $f(z)$ is restricted by the one of the following constraints

$$|f(z)| < \varepsilon(x) e^{c|y|} \quad \text{or} \quad |f(z)| < \frac{M e^{2|y|}}{|z|^\alpha}, \quad z = x + iy, \quad |z| \to \infty, \tag{3.4}$$

where $c < 2$, $\alpha > 1$ and $\varepsilon(x) \to 0$ for $x \to \infty$. Indeed, from the asymptotic expressions for Bessel functions it follows that

$$\left| \int_{a\pm ih}^{b\pm ih} [g(z) \pm f(z)]\, dz \right| = \left| \int_a^b \frac{\bar{H}_\nu^{(1,2)}(x \pm ih)}{\bar{J}_\nu(x \pm ih)} f(x \pm ih) dx \right| < \left\{ \begin{array}{l} M_1 e^{(c-2)h} \\ M_1'/h^\alpha \end{array} \right. \tag{3.5}$$

$$\left| \int_b^{b\pm i\infty} [g(z) \pm f(z)]\, dz \right| = \left| \int_0^\infty \frac{\bar{H}_\nu^{(1,2)}(b \pm ix)}{\bar{J}_\nu(b \pm ix)} f(b \pm ix) dx \right| < \left\{ \begin{array}{l} N_1 \varepsilon(b) \\ N_1'/b^{\alpha-1} \end{array} \right. \tag{3.6}$$

with constants M_1, M_1', N_1, N_1', and $H_\nu^{(1)}$ ($H_\nu^{(2)}$) corresponds to the upper (lower) sign.

Let us denote by $\lambda_{\nu,k} \neq 0$, $k = 1, 2, 3 \ldots$ the zeros of $\bar{J}_\nu(z)$ in the right half-plane, arranged in ascending order of the real part, $\mathrm{Re}\lambda_{\nu,k} \leq \mathrm{Re}\lambda_{\nu,k+1}$, (if some of these zeros lie on the imaginary axis we will take only zeros with positive imaginary part). All these zeros are simple. Note that for real $\nu > -1$ the function $\bar{J}_\nu(z)$ has only real zeros, except the case $A/B + \nu < 0$ when there are two purely imaginary zeros [7, 20]. By using the Wronskian $W[J_\nu(z), Y_\nu(z)] = 2/\pi z$ for (2.3) one finds

$$R[f(z), g(z)] = 2 \sum_k T_\nu(\lambda_{\nu,k}) f(\lambda_{\nu,k}) + r_{1\nu}[f(z)], \tag{3.7}$$

where we have introduced the notations

$$T_\nu(z) = \frac{z}{(z^2 - \nu^2) J_\nu^2(z) + z^2 J_\nu'^2(z)} \tag{3.8}$$

$$\begin{aligned} r_{1\nu}[f(z)] &= \pi i \sum_k \mathrm{Res}_{\mathrm{Im}z_k > 0} f(z) \frac{\bar{H}_\nu^{(1)}(z)}{\bar{J}_\nu(z)} - \pi i \sum_k \mathrm{Res}_{\mathrm{Im}z_k < 0} f(z) \frac{\bar{H}_\nu^{(2)}(z)}{\bar{J}_\nu(z)} - \\ &-\pi \sum_k \mathrm{Res}_{\mathrm{Im}z_k = 0} f(z) \frac{\bar{Y}_\nu(z)}{\bar{J}_\nu(z)}. \end{aligned} \tag{3.9}$$

Here z_k ($\neq \lambda_{\nu,i}$) are the poles for the function $f(z)$ in the region $Rez > a > 0$. Substituting (3.7) into (2.11) we obtain that for the function $f(z)$ meroporphic in the half-plane $\mathrm{Re}z \geq a$ and satisfying the condition (3.4) the following formula takes place

$$\begin{aligned} &\lim_{b \to +\infty} \left\{ 2 \sum_{k=m}^{n} T_\nu(\lambda_{\nu,k}) f(\lambda_{\nu,k}) + r_{1\nu}[f(z)] - \text{p.v.} \int_a^b f(x) dx \right\} = \\ &-\frac{1}{2} \int_a^{a+i\infty} f(z) \frac{\bar{H}_\nu^{(1)}(z)}{\bar{J}_\nu(z)} dz - \frac{1}{2} \int_a^{a-i\infty} f(z) \frac{\bar{H}_\nu^{(2)}(z)}{\bar{J}_\nu(z)} dz, \end{aligned} \tag{3.10}$$

where $\mathrm{Re}\lambda_{\nu,m-1} < a < \mathrm{Re}\lambda_{\nu,m}$, $\mathrm{Re}\lambda_{\nu,n} < b < \mathrm{Re}\lambda_{\nu,n+1}$, $a < \mathrm{Re}z_k < b$. We will apply this formula to the function $f(z)$ meromorphic in the half-plane $\mathrm{Re}z \geq 0$ taking $a \to 0$. Let us consider separately two cases.

3.1 Case (a)

Let $f(z)$ have no poles on the imaginary axis, except possibly at $z=0$, and

$$f(ze^{\pi i}) = -e^{2\nu\pi i}f(z) + o(z^{\beta_\nu}), \quad z \to 0 \tag{3.11}$$

(this condition is trivially satisfied for the function $f(z) = o(z^{\beta_\nu})$), with

$$\beta_\nu = \begin{cases} 2|\mathrm{Re}\nu| - 1 & \text{for integer } \nu \\ \mathrm{Re}\nu + |\mathrm{Re}\nu| - 1 & \text{for noninteger } \nu \end{cases} \tag{3.12}$$

Under this condition for values ν for which $\bar{J}_\nu(z)$ have no purely imaginary zeros the rhs of Eq.(3.10) in the limit $a \to 0$ can be presented in the form

$$-\frac{1}{\pi}\int_\rho^\infty \frac{\bar{K}_\nu(x)}{\bar{I}_\nu(x)}\left[e^{-\nu\pi i}f(xe^{\pi i/2}) + e^{\nu\pi i}f(xe^{-\pi i/2})\right]dx + \int_{\gamma_\rho^+} f(z)\frac{\bar{H}_\nu^{(1)}(z)}{\bar{J}_\nu(z)}dz - \int_{\gamma_\rho^-} f(z)\frac{\bar{H}_\nu^{(2)}(z)}{\bar{J}_\nu(z)}dz, \tag{3.13}$$

with γ_ρ^+ and γ_ρ^- being upper and lower halfs of the semicircle in the right half-plane with radius ρ and with center at point $z=0$, described in the positive sense with respect to this point. In (3.13) we have introduced modified Bessel functions $I_\nu(z)$ and $K_\nu(z)$ [21]. It follows from (3.11) that for $z \to 0$

$$\frac{\bar{H}_\nu^{(1)}(z)}{\bar{J}_\nu(z)}f(z) = \frac{\bar{H}_\nu^{(2)}(ze^{-\pi i})}{\bar{J}_\nu(ze^{-\pi i})}f(ze^{-\pi i}) + o(z^{-1}). \tag{3.14}$$

From here for $\rho \to 0$ one finds

$$D_\nu \equiv \int_{\gamma_\rho^+} f(z)\frac{\bar{H}_\nu^{(1)}(z)}{\bar{J}_\nu(z)}dz - \int_{\gamma_\rho^-} f(z)\frac{\bar{H}_\nu^{(2)}(z)}{\bar{J}_\nu(z)}dz = -\pi\mathrm{Res}_{z=0}f(z)\frac{\bar{Y}_\nu(z)}{\bar{J}_\nu(z)} \tag{3.15}$$

Indeed,

$$\begin{aligned} D_\nu &= \int_{\gamma_\rho^+} f(z)\frac{\bar{H}_\nu^{(1)}(z)}{\bar{J}_\nu(z)}dz + \int_{\gamma_{1\rho}^+} f(ze^{-\pi i})\frac{\bar{H}_\nu^{(2)}(ze^{-\pi i})}{\bar{J}_\nu(ze^{-\pi i})}dz = \int_{\gamma_\rho^+ + \gamma_{1\rho}^+} f(z)\frac{\bar{H}_\nu^{(1)}(z)}{\bar{J}_\nu(z)}dz + \\ &+ \int_{\gamma_{1\rho}^+} o(z^{-1})dz = i\int_{\gamma_\rho^+ + \gamma_{1\rho}^+} f(z)\frac{\bar{Y}_\nu(z)}{\bar{J}_\nu(z)}dz + \int_{\gamma_{1\rho}^+} o(z^{-1})dz, \end{aligned} \tag{3.16}$$

where $\gamma_{1\rho}^+$ ($\gamma_{1\rho}^-$, see below) is the upper (lower) half of the semicircle with radius ρ in the left half-plane with center at $z=0$ (described in the positive sense). In the last equality we have used the condition that integral p.v. $\int_0^b f(x)dx$ converges at lower limit. By similar way it can be seen that

$$D_\nu = i\int_{\gamma_\rho^- + \gamma_{1\rho}^-} f(z)\frac{\bar{Y}_\nu(z)}{\bar{J}_\nu(z)}dz + \int_{\gamma_{1\rho}^-} o(z^{-1})dz. \tag{3.17}$$

Combining the last two results we obtain (3.15) in the limit $\rho \to 0$. By using (3.10), (3.13) and (3.15) we have [11]:

Theorem 2. *If f(z) is a single valued analytic function in the half-plane* $\mathrm{Re}z \geq 0$ *(with possible branch point at* $z=0$*) except the poles* z_k $(\neq \lambda_{\nu,i})$, $\mathrm{Re}z_k > 0$ *(for the case of function* $f(z)$ *having purely imaginary poles see Remark after Theorem 3), and satisfy conditions (3.4) and (3.11), then in the case of* ν *for which the function* $\bar{J}_\nu(z)$ *has no purely imaginary zeros, the following formula is valid*

$$\begin{aligned} &\lim_{b\to+\infty}\left\{2\sum_{k=1}^n T_\nu(\lambda_{\nu,k})f(\lambda_{\nu,k}) + r_{1\nu}[f(z)] - \text{p.v.}\int_0^b f(x)dx\right\} = \\ &= \frac{\pi}{2}\mathrm{Res}_{z=0}f(z)\frac{\bar{Y}_\nu(z)}{\bar{J}_\nu(z)} - \frac{1}{\pi}\int_0^\infty \frac{\bar{K}_\nu(x)}{\bar{I}_\nu(x)}\left[e^{-\nu\pi i}f(xe^{\pi i/2}) + e^{\nu\pi i}f(xe^{-\pi i/2})\right]dx, \end{aligned} \tag{3.18}$$

where on the left $\mathrm{Re}\lambda_{\nu,n} < b < \mathrm{Re}\lambda_{\nu,n+1}$, $0 < \mathrm{Re}z_k < b$, *and* $T_\nu(\lambda_{\nu,k})$ *and* $r_{1\nu}[f(z)]$ *are determined by relations (3.8) and (3.9).*

Under the condition (3.11) the integral on the right converges at lower limit. Recall that we assume the existence of the integral on the left as well (see section 2). The formula (3.18) and analog ones given below are especially useful for numerical calculations of the sums over $\lambda_{\nu,k}$ as under the first conditions in (3.4) the integral on the right converges exponentially fast at the upper limit.

Remark. Deriving the formula (3.18) we have assumed that the function $f(z)$ is meromorphic in the half-plane $\mathrm{Re}z \geq 0$ (except possibly at $z = 0$). However this formula is valid also for some functions having branch points on the imaginary axis, for example,

$$f(z) = f_1(z) \prod_{l=1}^{k} \left(z^2 + c_l^2\right)^{\pm 1/2}, \tag{3.19}$$

with meromorphic function $f_1(z)$. The proof for (3.18) in this case is similar to the given above with difference that branch points $\pm ic_l$ have to be around on the right along contours with small radii. In view of the further applications to the Casimir effect (see below) let us consider the case $k = 1$. By taking into account that

$$\left(z^2 + c^2\right)^{1/2} = \begin{cases} \left|z^2 + c^2\right|^{1/2} & \text{if} \quad |z| < c \\ \left|z^2 + c^2\right|^{1/2} e^{i\pi/2} & \text{if} \quad \mathrm{Im}z > c \\ \left|z^2 + c^2\right|^{1/2} e^{-i\pi/2} & \text{if} \quad \mathrm{Im}z < -c \end{cases} \tag{3.20}$$

from (3.18) one obtains

$$\begin{aligned} \lim_{b \to +\infty} \left\{ 2 \sum_{k=1}^{n} T_\nu(\lambda_{\nu,k}) f(\lambda_{\nu,k}) + r_{1\nu}[f(z)] - \text{p.v.} \int_0^b f(x) dx \right\} = \frac{\pi}{2} \mathrm{Res}_{z=0} f(z) \frac{\bar{Y}_\nu(z)}{\bar{J}_\nu(z)} - \\ - \frac{1}{\pi} \int_0^c \frac{\bar{K}_\nu(x)}{\bar{I}_\nu(x)} \left[e^{-\nu\pi i} f_1(x e^{\pi i/2}) + e^{\nu\pi i} f_1(x e^{-\pi i/2}) \right] \left(c^2 - x^2\right)^{\pm 1/2} dx \mp \\ \mp \frac{i}{\pi} \int_c^\infty \frac{\bar{K}_\nu(x)}{\bar{I}_\nu(x)} \left[e^{-\nu\pi i} f_1(x e^{\pi i/2}) - e^{\nu\pi i} f_1(x e^{-\pi i/2}) \right] \left(x^2 - c^2\right)^{\pm 1/2} dx, \end{aligned} \tag{3.21}$$

where $f(z) = f_1(z)\left(z^2 + c^2\right)^{\pm 1/2}$, $c > 0$. In Section 11 we apply this formula with analytic function $f_1(z)$ to derive the expressions for the regularized values of the energy-momentum tensor components in the region inside the perfectly conducting cylindrical shell. ■

For an analitic function $f(z)$ the formula (3.18) yields

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{2\lambda_{\nu,k} f(\lambda_{\nu,k})}{\left(\lambda_{\nu,k}^2 - \nu^2\right) J_\nu^2(\lambda_{\nu,k}) + \lambda_{\nu,k}^2 J_\nu'^2(\lambda_{\nu,k})} = \int_0^\infty f(x) dx + \frac{\pi}{2} \mathrm{Res}_{z=0} f(z) \frac{\bar{Y}_\nu(z)}{\bar{J}_\nu(z)} - \\ - \frac{1}{\pi} \int_0^\infty \frac{\bar{K}_\nu(x)}{\bar{I}_\nu(x)} \left[e^{-\nu\pi i} f(x e^{\pi i/2}) + e^{\nu\pi i} f(x e^{-\pi i/2}) \right] dx. \end{aligned} \tag{3.22}$$

By taking in this formula $\nu = 1/2$, $A = 1$, $B = 0$ (see the notation (3.1)) as a particular case we immediately receive the APF in the form (2.13). In like manner substituting $\nu = 1/2$, $A = 1$, $B = 2$ we obtain APF in the form (2.15). Consequently the formula (3.18) is a generalization of APF for general ν (with restrictions given above) and for functions $f(z)$ having poles in the right half-plane.

Having in mind the further applications to the Casimir effect in Sections 8 and 11 let us choose in (3.22)

$$f(z) = F(z)J^2_{\nu+m}(zt), \quad t > 0, \quad \mathrm{Re}\nu \geq 0 \tag{3.23}$$

with m being an integer. Now the conditions (3.4) formulated in terms of $F(z)$ are in form

$$|F(z)| < |z|\varepsilon e^{(c-2t)|y|} \quad \text{or} \quad |f(z)| < \frac{Me^{2(1-t)|y|}}{|z|^{\alpha-1}}, \quad z = x + iy, \quad |z| \to \infty \tag{3.24}$$

with the same notations as in (3.4). In like manner from the condition (3.11) for $F(z)$ one has

$$F(ze^{\pi i}) = -F(z) + o(z^{-2m-1}), \quad z \to 0. \tag{3.25}$$

Now as a consequence of (3.22) we obtain that if the conditions (3.24) and (3.25) are satisfied, then for the function $F(z)$ analytic in the right half-plane, the following formula takes place

$$2\sum_{k=1}^{\infty} T_\nu(\lambda_{\nu,k})F(\lambda_{\nu,k})J^2_{\nu+m}(\lambda_{\nu,k}t) = \int_0^\infty F(x)J^2_{\nu+m}(xt)dx - \\ -\frac{1}{\pi}\int_0^\infty \frac{\bar{K}_\nu(x)}{\bar{I}_\nu(x)} I^2_{\nu+m}(xt)\left[F(xe^{\pi i/2}) + F(xe^{-\pi i/2})\right]dx \tag{3.26}$$

for $\mathrm{Re}\nu \geq 0$ and $\mathrm{Re}\nu + m \geq 0$.

3.2 Case (b)

Let $f(z)$ be a function satisfying the condition

$$f(xe^{\pi i/2}) = -e^{2\nu\pi i}f(xe^{-\pi i/2}) \tag{3.27}$$

for real x. It is clear that if $f(z)$ have purely imaginary poles, then they are complex conjugate: $\pm iy_k$, $y_k > 0$. By (3.27) the rhs of (3.10) for $a \to 0$ and $\arg\lambda_{\nu,k} = \pi/2$ may be written as

$$\left(\int_{\gamma_\rho^+} + \sum_{\sigma_k = iy_k, \lambda_{\nu,k}} \int_{C_\rho(\sigma_k)}\right) \frac{\bar{H}^{(1)}_\nu(z)}{\bar{J}_\nu(z)} f(z)dz - \left(\int_{\gamma_\rho^-} + \sum_{\sigma_k = -iy_k, -\lambda_{\nu,k}} \int_{C_\rho(\sigma_k)}\right) \frac{\bar{H}^{(2)}_\nu(z)}{\bar{J}_\nu(z)} f(z)dz, \tag{3.28}$$

where $C_\rho(\sigma_k)$ denotes the right half of the circle with center at the point σ_k and radius ρ, described in the positive sense, and the contours $\gamma_\rho^\pm$ are the same as in (3.13). We have used the fact the purely imaginary zeros of $\bar{J}_\nu(z)$ are complex conjugate numbers, as $\bar{J}_\nu(ze^{\pi i}) = e^{\nu\pi i}\bar{J}_\nu(z)$. We have used also the fact that on the right of (3.10) the integrals (with $a = 0$) along straight segments of the upper and lower imaginary semiaxes are canceled, as in accordance of (3.27) for $\arg z = \pi/2$

$$\frac{\bar{H}^{(1)}_\nu(z)}{\bar{J}_\nu(z)} f(z) = \frac{\bar{H}^{(2)}_\nu(ze^{-\pi i})}{\bar{J}_\nu(ze^{-\pi i})} f(ze^{-\pi i}). \tag{3.29}$$

Let us show that from (3.29) for $z_0 = x_0e^{\pi i/2}$ it follows that this relation is valid for any z in a small enough region including this point. Namely, as the function $f(z)\bar{H}^{(p)}_\nu(z)/\bar{J}_\nu(z)$, $p = 1, 2$ is meromorphic near the point $(-1)^{p+1}x_0e^{\pi i/2}$, there exists a neighbourhood of this point where this function is presented as a Laurent expansion

$$\frac{\bar{H}^{(p)}_\nu(z)}{\bar{J}_\nu(z)} f(z) = \sum_{n=-n_0}^{\infty} \frac{a_n^{(p)}}{\left[z - (-1)^{p+1}x_0e^{\pi i/2}\right]^n}. \tag{3.30}$$

From (3.29) for $z = xe^{\pi i/2}$ one concludes

$$\sum_{n=-n_0}^{\infty} \frac{a_n^{(1)} e^{-n\pi i/2}}{(x-x_0)^n} = \sum_{n=-n_0}^{\infty} \frac{(-1)^n a_n^{(2)} e^{-n\pi i/2}}{(x-x_0)^n}, \tag{3.31}$$

and hence $a_n^{(1)} = (-1)^n a_n^{(2)}$. Our statement follows directly from here. By this it can be seen that

$$\int_{C_\rho(\sigma_k)} \frac{\bar{H}_\nu^{(1)}(z)}{\bar{J}_\nu(z)} f(z)dz - \int_{C_\rho(-\sigma_k)} \frac{\bar{H}_\nu^{(2)}(z)}{\bar{J}_\nu(z)} f(z)dz = 2\pi i \mathrm{Res}_{z=\sigma_k} \frac{\bar{H}_\nu^{(1)}(z)}{\bar{J}_\nu(z)} f(z) \tag{3.32}$$

where $\sigma_k = iy_k$, $\lambda_{\nu,k}$, $\arg\lambda_{\nu,k} = \pi/2$. Now by taking into account (3.15) and letting $\rho \to 0$ we get [11, 12]:

Theorem 3. *Let $f(z)$ be meromorphic function in the half-plane* $\mathrm{Re}z \geq 0$ *(except possibly at $z=0$) with poles z_k,* $\mathrm{Re}z_k > 0$ *and $\pm iy_k$, $y_k > 0$, $k = 1, 2, ...$ ($\neq \lambda_{\nu,p}$). If this function satisfy the conditions (3.4) and (3.27) then*

$$\lim_{b\to+\infty} \left\{ 2\sum_{k=1}^{n} T_\nu(\lambda_{\nu,k}) f(\lambda_{\nu,k}) + r_{1\nu}[f(z)] - \text{p.v.} \int_0^b f(x)dx \right\} =$$
$$= -\frac{\pi i}{2} \sum_{\eta_k=0, iy_k} (2-\delta_{0\eta_k}) \,\mathrm{Res}_{z=\eta_k} f(z) \frac{\bar{H}_\nu^{(1)}(z)}{\bar{J}_\nu(z)}, \tag{3.33}$$

where on the left $0 < \mathrm{Re}z_k < b$, $\mathrm{Re}\lambda_{\nu,n} < b < \mathrm{Re}\lambda_{\nu,n+1}$ *and $r_{1\nu}$ is defined by (3.9).*

Note that the residue terms in (3.32) with $\sigma_k = \lambda_{\nu,k}$, $\arg\lambda_{\nu,k} = \pi/2$ are equal to $4T_\nu(\lambda_{\nu,k})f(\lambda_{\nu,k})$ and are included in the first sum on the left of (3.33).

Remark. Let $\pm iy_k$, $y_k > 0$ and $\pm\lambda_{\nu,k}$, $\arg\lambda_{\nu,k} = \pi/2$ are purely imaginary poles of function $f(z)$ and purely imaginary zeros of $\bar{J}_\nu(z)$, correspondingly. Let function $f(z)$ satisfy condition

$$f(z) = -e^{2\nu\pi i} f(ze^{-\pi i}) + o\left((z-\sigma_k)^{-1}\right), \quad z \to \sigma_k, \quad \sigma_k = iy_k, \lambda_{\nu,k}. \tag{3.34}$$

Now in the limit $a \to 0$ the rhs of (3.10) can be presented in the form (3.28) plus integrals along the straight segments of the imaginary axis between the poles. Using the arguments similar those given above we obtain the relation (3.32) with additional contribution from the last term on the right of (3.34) in the form $\int_{C_\rho(-\sigma_k)} o\left((z-\sigma_k)^{-1}\right)dz$. In the limit $\rho \to 0$ the latter vanishes and sum of the integrals along the straight segments of the imaginary axis gives the principal value of the integral on the right of (3.18). As a result the formula (3.18) can be generalized for functions having purely imaginary poles and satisfying condition (3.34) writing instead of residue term on the right the sum of residues from the right of (3.33) and taking the principle value of the integral on the right. The latter exists due to the condition (3.34). ■

It follows from (3.33) an interesting result. Let $\lambda_{\mu,k}^{(1)}$ be zeros of the function $A_1 J_\mu(z) + B_1 z J_\mu'(z)$ with some real constants A_1 and B_1. Let $f(z)$ be an analytic function in the right half-plane satisfying condition (3.27) and $f(z) = o(z^\beta)$ for $z \to 0$, where $\beta = \max(\beta_\mu, \beta_\nu)$ (the definition β_ν see (3.12)). For this function from (3.33) we get

$$\sum_{k=1}^{\infty} T_\mu(\lambda_{\mu,k}^{(1)}) f(\lambda_{\mu,k}^{(1)}) = \sum_{k=1}^{\infty} T_\nu(\lambda_{\nu,k}) f(\lambda_{\nu,k}), \quad \mu = \nu + m\,. \tag{3.35}$$

For the case of Fourier-Bessel and Dini series this result is given in [20].

Let us consider some applications of the formula (3.33) to the special types of series. Firstly we choose in this formula

$$f(z) = F_1(z)J_\mu(zt), \quad t > 0, \tag{3.36}$$

where the function $F_2(z)$ is meromorphic on the right half-plane and satisfy conditions

$$|F_1(z)| < \varepsilon_1(x)e^{(c-t)|y|} \quad \text{or} \quad |F_1(z)| < M|z|^{-\alpha_1}e^{(2-t)|y|}, \quad |z| \to \infty, \tag{3.37}$$

with $c < 2$, $\alpha_1 > 1/2$, $\varepsilon_1(x) = o(\sqrt{x})$ for $x \to +\infty$, and condition

$$F_1(xe^{\pi i/2}) = -e^{(2\nu-\mu)\pi i}F_1(xe^{-\pi i/2}). \tag{3.38}$$

From (3.37) it follows that the integral p.v. $\int_0^\infty F_1(x)J_\mu(xt)dx$ converges at the upper limit and hence in this case the formula (3.33) may be written in the form

$$\begin{aligned}\sum_{k=1}^\infty T_\nu(\lambda_{\nu,k})F_1(\lambda_{\nu,k})J_\mu(\lambda_{\nu,k}t) &= \frac{1}{2}\text{p.v.}\int_0^\infty F_1(x)J_\mu(xt)dx - \frac{1}{2}r_{1\nu}\left[F_1(z)J_\mu(zt)\right] - \\ &- \frac{\pi i}{4}\sum_{\eta_k=0,iy_k}(2-\delta_{0\eta_k})\,\text{Res}_{z=\eta_k}F_2(z)J_\mu(zt)\frac{\bar{H}_\nu^{(1)}(z)}{\bar{J}_\nu(z)},\end{aligned} \tag{3.39}$$

For example, it follows from here that for $t < 1$, Reσ, Re$\nu > -1$

$$\sum_{k=1}^\infty \frac{T_\nu(\lambda_{\nu,k})}{\lambda_{\nu,k}^\sigma}J_{\sigma+\nu+1}(\lambda_{\nu,k})J_\nu(\lambda_{\nu,k}t) = \frac{1}{2}\int_0^\infty J_{\sigma+\nu+1}(z)J_\nu(zt)\frac{dz}{z^\sigma} = \frac{\left(1-t^2\right)^\sigma t^\nu}{2^{\sigma+1}\Gamma(\sigma+1)} \tag{3.40}$$

(for the value of integral see, e.g., [20]). For $B = 0$ this result is given in [23]. In a similar manner taking $\mu = \nu + m$,

$$F_1(z) = z^{\nu+m+1}\frac{J_\sigma(a\sqrt{z^2+z_1^2})}{\left(z^2+z_1^2\right)^{\sigma/2}}, \quad a > 0 \tag{3.41}$$

with Re$\nu \geq 0$ and Re$\nu + m \geq 0$, from (3.39) for $a < 2 - t$, Re$\sigma >$ Re$\nu + m$ one finds

$$\begin{aligned}&\sum_{k=1}^\infty T_\nu(\lambda_{\nu,k})\lambda_{\nu,k}^{\nu+m+1}J_{\nu+m}(\lambda_{\nu,k}t)\frac{J_\sigma(a\sqrt{\lambda_{\nu,k}^2+z_1^2})}{\left(\lambda_{\nu,k}^2+z_1^2\right)^{\sigma/2}} = \\ &= \frac{1}{2}\int_0^\infty x^{\nu+m+1}J_{\nu+m}(xt)\frac{J_\sigma(a\sqrt{x^2+z_1^2})}{\left(x^2+z_1^2\right)^{\sigma/2}}dx = \frac{t^{\nu+1}}{a^\sigma}\left(-z_1\right)^{m+1}\frac{J_{m+1}(z_1\sqrt{a^2-t^2})}{\left(a^2-t^2\right)^{(m+1)/2}},\ a > t\end{aligned} \tag{3.42}$$

and the sum is zero when $a < t$. Here we have used the known value for Sonine integral [20].

If an addition to (3.37), (3.38) the function F_1 satisfies conditions

$$F_1(xe^{\pi i/2}) = -e^{\mu\pi i}F_1(xe^{-\pi i/2}) \tag{3.43}$$

and

$$|F_2(z)| < \varepsilon_1(x)e^{c_1 t|y|} \quad \text{or} \quad |F_2(z)| < M|z|^{-\alpha_1}e^{t|y|}, \quad |z| \to \infty, \tag{3.44}$$

when the formula (5.7)(see below) with $B = 0$ may be applied to the integral on the right of (3.39). This gives [11, 12]

Corollary 1. *Let $F(z)$ be meromorphic function in the half-plane* $\mathrm{Re}z \geq 0$ *(except possibly at $z=0$) with poles z_k,* $\mathrm{Re}z_k > 0$ *and $\pm iy_k$, $y_k > 0$ ($\neq \lambda_{\nu,i}$). If $F(z)$ satisfy condition*

$$F(xe^{\pi i/2}) = (-1)^{m+1}e^{\nu\pi i}F(xe^{-\pi i/2}) \tag{3.45}$$

with an integer m, and to one of inequalities

$$|F(z)| < \varepsilon_1(x)e^{a|y|} \quad or \quad |F(z)| < M|z|^{-\alpha_1}e^{a_0|y|}, \quad |z| \to \infty, \tag{3.46}$$

with $a < \min(t, 2-t) \equiv a_0$, $\varepsilon_1(x) = o(x^{1/2})$, $x \to +\infty$, $\alpha_1 > 1/2$, the following formula is valid

$$\begin{aligned} &\sum_{k=1}^{\infty} T_\nu(\lambda_{\nu,k})F(\lambda_{\nu,k})J_{\nu+m}(\lambda_{\nu,k}t) = \\ &= \frac{\pi i}{4} \sum_{\eta_k=0,iy_k,z_k} (2-\delta_{0\eta_k}) \,\mathrm{Res}_{z=\eta_k} \left\{ \left[J_{\nu+m}(zt)\bar{Y}_\nu(z) - Y_{\nu+m}(zt)\bar{J}_\nu(z)\right] \frac{F(z)}{\bar{J}_\nu(z)} \right\}. \end{aligned} \tag{3.47}$$

Recall that for the imaginary zeros $\lambda_{\nu,k}$, in lhs of (3.47) the zeros with positive imaginary parts enter only. By using the formula (3.47) a number of Furier-Bessel and Dini series can be summarized (see, for instance, below).

Remark. The formula (3.47) may be obtained also by considering the integral

$$\frac{1}{\pi} \int_{C_h} \left[H^{(2)}_{\nu+m}(zt)\bar{H}^{(1)}_\nu(z) - H^{(1)}_{\nu+m}(zt)\bar{H}^{(2)}_\nu(z)\right] \frac{F(z)}{\bar{J}_\nu(z)} dz, \tag{3.48}$$

where C_h is an rectangle with vertices $\pm ih$, $b \pm ih$, described in the positive sense (purely imaginary poles of $F(z)/\bar{J}_\nu(z)$ and the origin are around by semicircles in the right half-plane with small radii). This integral is equal to the sum of residues over the poles within C_h (points z_k, $\lambda_{\nu,k}$, (Rez_k, $\mathrm{Re}\lambda_{\nu,k} > 0$)). On the other hand it follows from (3.45) that integrals along the segments of the imaginary axes cancel each other. The sum of integrals along the conjugate semicircles give the sum of residues over purely imaginary poles in the upper half plane. The integrals along the remained three segments of C_h in accordance with (3.46) approach to zero in the limit b, $h \to \infty$. Equating these expressions for (3.48) one immediately obtains the result (3.47). ■

From (3.47) for $t=1$, $F(z) = J_\nu(zx)$, $m=1$ one obtains [7, 20]

$$\sum_{k=1}^{\infty} T_\nu(\lambda_{\nu,k})J_\nu(\lambda_{\nu,k}x)J_{\nu+1}(\lambda_{\nu,k}) = \frac{x^\nu}{2}, \quad 0 \leq x < 1. \tag{3.49}$$

By similar way choosing $m=0$, $F(z) = zJ_\nu(zx)/\left(z^2-a^2\right)$, $B=0$ we obtain the Kneser-Sommerfeld expansion [20]:

$$\sum_{k=1}^{\infty} \frac{J_\nu(\lambda_{\nu,k}t)J_\nu(\lambda_{\nu,k}x)}{\left(\lambda^2_{\nu,k}-a^2\right)J^2_{\nu+1}(\lambda_{\nu,k})} = \frac{\pi}{4}\frac{J_\nu(ax)}{J_\nu(a)}\left[J_\nu(at)Y_\nu(a) - Y_\nu(at)J_\nu(a)\right], \quad 0 \leq x \leq t \leq 1. \tag{3.50}$$

In (3.47) as a function $F(z)$ one may choose, for example, the following functions

$$z^{\rho-1}\prod_{l=1}^{n}\left(z^2+z_l^2\right)^{-\mu_l/2} J_{\mu_l}(b_l\sqrt{z^2+z_l^2}), \tag{3.51}$$

$$\text{for} \quad \mathrm{Re}\nu < \sum_{l=1}^{n} \mathrm{Re}\mu_l + n/2 + 2p + 3/2 - m - \delta_{ba_0}, \quad b \le a_0,\, b = \sum_{l=1}^{n} b_l;$$

$$z^{\rho-2n-1} \prod_{l=1}^{n} \left[1 - J_0(b_l z)\right], \tag{3.52}$$

$$\text{for} \quad \mathrm{Re}\nu < 2n + 2p + 3/2 - m - \delta_{ba_0};$$

$$z^{\rho-1} \prod_{l=1}^{n} \left(z^2 + z_l^2\right)^{\mu_l/2} Y_{\mu_l}\left(b_l\sqrt{z^2 + z_l^2}\right),\ \mu_l > 0\text{-half of an odd integer}, \tag{3.53}$$

$$\text{for} \quad \mathrm{Re}\nu < -\sum_{l=1}^{n} \mu_l + n/2 + 2p + 3/2 - m - \delta_{ba_0};$$

$$z^{\rho-1} \prod_{l=1}^{n} z^{|k_l|} \left[J_{\mu_l+k_l}(a_l z) Y_{\mu_l}(b_l z) - Y_{\mu_l+k_l}(a_l z) J_{\mu_l}(b_l z)\right], \quad k_l\text{- integer}, \tag{3.54}$$

$$\text{for} \quad \mathrm{Re}\nu < n + 2p + 3/2 - m - \sum |k_l| - \delta_{\tilde{a},a_0}, \quad \tilde{a} \equiv \sum_{l=1}^{n} |a_l - b_l| \le a_0;$$

with $\rho = \nu + m - 2p$ (p - integer), as well as any products between these functions and with $\prod_l \left(z^2 - c_l^2\right)^{-p_l}$, provided the condition (3.46) is satisfied. For example, the following formulae take place

$$\begin{aligned} &\sum_{k=1}^{\infty} j_{\nu,k}^{\nu-2} \frac{J_\nu(j_{\nu,k}t)}{J_{\nu+1}^2(j_{\nu,k})} \prod_{l=1}^{n} \left[J_{\mu_l}(a_l j_{\nu,k}) Y_{\mu_l}(b_l j_{\nu,k}) - Y_{\mu_l}(a_l j_{\nu,k}) J_{\mu_l}(b_l j_{\nu,k})\right] = \\ &\quad = \frac{2^{\nu-2}}{\pi^n t^\nu}\left(1 - t^{2\nu}\right) \prod_{l=1}^{n} \frac{b_l^{\mu_l}}{\mu_l a_l^{\mu_l}} \left[1 - \left(\frac{a_l}{b_l}\right)^{2\mu_l}\right], \quad 0 < t \le 1, \end{aligned} \tag{3.55}$$

$$c \equiv \sum_{l=1}^{n} |a_l - b_l| \le t, \quad a_l, b_l > 0, \quad \mathrm{Re}\mu_l \ge 0, \quad \mathrm{Re}\nu < n + 3/2 - \delta_{ct};$$

$$\sum_{k=1}^{\infty} \frac{J_\nu(j_{\nu,k}t) J_{\nu+1}(\lambda j_{\nu,k})}{j_{\nu,k}^{2n+3} J_{\nu+1}^2(j_{\nu,k})} \prod_{l=1}^{n} \left[1 - J_0(b_l j_{\nu,k})\right] = \frac{\lambda^{\nu+1}\left(1 - t^{2\nu}\right)}{4^{n+1}\nu(\nu+1)t^\nu} \prod_{l=1}^{n} b_l^2, \tag{3.56}$$

$$\lambda + \sum_{l=1}^{n} b_l \le t \le 1, \quad \lambda, b_l > 0;$$

$$\sum_{k=1}^{\infty} \frac{J_\mu(j_{\nu,k}b) J_{\nu+1}(\lambda j_{\nu,k}) J_\nu(j_{\nu,k}t)}{\left(j_{\nu,k}^2 - a^2\right) j_{\nu,k}^{\mu+1} J_{\nu+1}^2(j_{\nu,k})} = \frac{\pi J_{\nu+1}(a\lambda)}{4a^{\mu+1}} \frac{J_\mu(ba)}{J_\nu(a)} \left[Y_\nu(a) J_\nu(at) - J_\nu(a) Y_\nu(at)\right], \tag{3.57}$$

$$\lambda + b \le t \le 1,\ \lambda, b > 0,\ \mathrm{Re}\mu > -7/2 + \delta_{\lambda+b,t},$$

where $j_{\nu,k}$ are zeros of $J_\nu(z)$.

The examples of the series over zeros of Bessel functions we found in literature (see, e.g., [7, 20, 23, 22]), when the corresponding sum was evaluated in finite terms, are particular cases of the formulae given in this section.

4 Summation formulae over zeros of $\bar{J}_\nu(z)\bar{Y}_\nu(\lambda z) - \bar{Y}_\nu(z)\bar{J}_\nu(\lambda z)$

In this section we will consider the series over zeros of the function

$$C_\nu^{AB}(\lambda, z) \equiv \bar{J}_\nu(z)\bar{Y}_\nu(\lambda z) - \bar{Y}_\nu(z)\bar{J}_\nu(\lambda z), \tag{4.1}$$

where the bared quantities are defined as (3.1). Series of this type arise in calculations of the vacuum expectation values in confined regions with boundaries of spherical and cylindrical form.

To obtain a summation formula for these series let us substitute in (2.11)

$$g(z)=\frac{1}{2i}\left[\frac{\bar{H}_{\nu}^{(1)}(\lambda z)}{\bar{H}_{\nu}^{(1)}(z)}+\frac{\bar{H}_{\nu}^{(2)}(\lambda z)}{\bar{H}_{\nu}^{(2)}(z)}\right]\frac{h(z)}{C_{\nu}^{AB}(\lambda,z)},\quad f(z)=\frac{h(z)}{\bar{H}_{\nu}^{(1)}(z)\bar{H}_{\nu}^{(2)}(z)},\tag{4.2}$$

where for definiteness we shall assume that $\lambda>1$. The sum and difference of these functions are

$$g(z)-(-1)^{k}f(z)=-i\frac{\bar{H}_{\nu}^{(k)}(\lambda z)}{\bar{H}_{\nu}^{(k)}(z)}\frac{h(z)}{C_{\nu}^{AB}(\lambda,z)},\quad k=1,2.\tag{4.3}$$

The conditions for GAPF written in terms of the function $h(z)$ are as follows

$$|h(z)|<\varepsilon_{2}(x)e^{c_{2}|y|}\quad\text{or}\quad|h(z)|<M|z|^{-\alpha_{2}}e^{2(\lambda-1)|y|},\quad|z|\to\infty,\quad z=x+iy\tag{4.4}$$

where $c_2<2(\lambda-1)$, $x^{2\delta_{B0}-1}\varepsilon_2(x)\to 0$ for $x\to+\infty$, $\alpha_2>2\delta_{B0}$. Let $\gamma_{\nu,k}$ be zeros for the function $C_{\nu}^{AB}(\lambda,z)$ in the right half-plane. In this section we will assume values of ν, A, B for which all these zeros are real and simple, and the function $\bar{H}_{\nu}^{(1)}(z)$ ($\bar{H}_{\nu}^{(2)}(z)$) has no zeros in the right half of the upper (lower) half-plane. As we will see later these conditions are satisfied in physical problems considered below. For real ν, A, B the zeros $\gamma_{\nu,k}$ are simple. To see this note that the function $J_{\nu}(tz)\bar{Y}_{\nu}(z)-Y_{\nu}(tz)\bar{J}_{\nu}(z)$ is cylinder function with respect to t. Using the standard result for indefinite integrals containing any two cylinder functions (see [20, 21]) it can be seen that

$$\int_{1}^{\lambda}t\left[J_{\nu}(tz)\bar{Y}_{\nu}(z)-Y_{\nu}(tz)\bar{J}_{\nu}(z)\right]^{2}dt=\frac{2}{\pi^{2}zT_{\nu}^{AB}(\lambda,z)},\quad z=\gamma_{\nu,k},\tag{4.5}$$

where we have introduced the notation

$$T_{\nu}^{AB}(\lambda,z)=z\left\{\frac{\bar{J}_{\nu}^{2}(z)}{\bar{J}_{\nu}^{2}(\lambda z)}\left[A^{2}+B^{2}(\lambda^{2}z^{2}-\nu^{2})\right]-A^{2}-B^{2}(z^{2}-\nu^{2})\right\}^{-1}.\tag{4.6}$$

On the other hand

$$\frac{\partial}{\partial z}C_{\nu}^{AB}(\lambda,z)=\frac{2}{\pi T_{\nu}^{AB}(\lambda,z)}\frac{\bar{J}_{\nu}(\lambda z)}{\bar{J}_{\nu}(z)},\quad z=\gamma_{\nu,k}.\tag{4.7}$$

Combining the last two results we deduce that for real ν, A, B the derivative (4.7) is nonzero and hence the zeros $z=\gamma_{\nu,k}$ are simple. By using this it can be seen that

$$\mathrm{Res}_{z=\gamma_{\nu,k}}g(z)=\frac{\pi}{2i}T_{\nu}^{AB}(\lambda,\gamma_{\nu,k}).\tag{4.8}$$

Hence if the function $h(z)$ is analytic in the half-plane $\mathrm{Re}z\geq a>0$ except at the poles z_k ($\neq\gamma_{\nu,i}$) and satisfy to the one of two conditions (4.4), the following formula takes place

$$\lim_{b\to+\infty}\left\{\frac{\pi^{2}}{2}\sum_{k=n}^{m}T_{\nu}^{AB}(\lambda,\gamma_{\nu,k})h(\gamma_{\nu,k})+r_{2\nu}[h(z)]-\text{p.v.}\int_{a}^{b}\frac{h(x)dx}{\bar{J}_{\nu}^{2}(x)+\bar{Y}_{\nu}^{2}(kx)}\right\}=$$
$$=\frac{i}{2}\int_{a}^{a+i\infty}\frac{\bar{H}_{\nu}^{(1)}(\lambda z)}{\bar{H}_{\nu}^{(1)}(z)}\frac{h(z)}{C_{\nu}^{AB}(\lambda,z)}dz-\frac{i}{2}\int_{a}^{a-i\infty}\frac{\bar{H}_{\nu}^{(2)}(\lambda z)}{\bar{H}_{\nu}^{(2)}(z)}\frac{h(z)}{C_{\nu}^{AB}(\lambda,z)}dz.\tag{4.9}$$

Here we assumed that the integral on the left exists, $\gamma_{\nu,n-1}<a<\gamma_{\nu,n}$, $\gamma_{\nu,m}<b<\gamma_{\nu,m+1}$, $a<\mathrm{Re}z_k<b$, $\mathrm{Re}z_k\leq\mathrm{Re}z_{k+1}$, and the following notation is introduced

$$r_{2\nu}[h(z)]=\pi\sum_{k}\mathrm{Res}_{\mathrm{Im}z_k=0}\left[\frac{\bar{J}_{\nu}(z)\bar{J}_{\nu}(\lambda z)+\bar{Y}_{\nu}(z)\bar{Y}_{\nu}(\lambda z)}{\bar{J}_{\nu}^{2}(z)+\bar{Y}_{\nu}^{2}(z)}\frac{h(z)}{C_{\nu}^{AB}(\lambda,z)}\right]+$$
$$+\pi\sum_{k,l=1,2}\mathrm{Res}_{(-1)^{l}\mathrm{Im}z_k<0}\left[\frac{\bar{H}_{\nu}^{(l)}(\lambda z)}{\bar{H}_{\nu}^{(l)}(z)}\frac{h(z)}{C_{\nu}^{AB}(\lambda,z)}\right].\tag{4.10}$$

The general formula (4.9) is a direct consequence of GAPF and will be as starting point for the further applications in this section. In the limit $a \to 0$ one has [11, 12]:

Corollary 2. *Let $h(z)$ be analytic function for* $\mathrm{Re}z \geq 0$ *except the poles z_k ($\neq \gamma_{\nu i}$),* $\mathrm{Re}z_k > 0$ *(with possible branch point $z = 0$). If it satisfies one of two conditions (4.4) and*

$$h(ze^{\pi i}) = -h(z) + o(z^{-1}), \quad z \to 0, \tag{4.11}$$

and the integral

$$\text{p.v.} \int_a^b \frac{h(x)dx}{\bar{J}_\nu^2(x) + \bar{Y}_\nu^2(x)} \tag{4.12}$$

exists, then

$$\lim_{b\to+\infty} \left\{ \frac{\pi^2}{2} \sum_{k=1}^{m} h(\gamma_{\nu,k}) T_\nu^{AB}(\lambda, \gamma_{\nu,k}) + r_{3\nu}[h(z)] - \text{p.v.} \int_0^b \frac{h(x)dx}{\bar{J}_\nu^2(x) + \bar{Y}_\nu^2(x)} \right\} =$$
$$= -\frac{\pi}{2} \mathrm{Res}_{z=0} \left[\frac{h(z)\bar{H}_\nu^{(1)}(\lambda z)}{C_\nu^{AB}(\lambda, z)\bar{H}_\nu^{(1)}(z)} \right] - \frac{\pi}{4} \int_0^\infty \frac{\bar{K}_\nu(\lambda x)}{\bar{K}_\nu(x)} \frac{\left[h(xe^{\pi i/2}) + h(xe^{-\pi i/2})\right] dx}{\bar{K}_\nu(x)\bar{I}_\nu(\lambda x) - \bar{K}_\nu(\lambda x)\bar{I}_\nu(x)} \tag{4.13}$$

In the following we shall use this formula to derive the regularized vacuum energy momentum-tensor for the region between two spherical and cylindrical surfaces. Note that (4.13) may be generalized for the functions $h(z)$ with purely imaginary poles $\pm iy_k$, $y_k > 0$ satisfying condition

$$h(ze^{\pi i}) = -h(z) + o\left((z \mp iy_k)^{-1}\right), \quad z \to \pm iy_k. \tag{4.14}$$

The corresponding formula is obtained from (4.13) by adding residue terms for $z = iy_k$ in the form of (4.16) (see below) and taking the principal value of the integral on the right. The arguments here are similar to those for Remark after Theorem 3.

By the way similar to (3.18) one has another result [11, 12]:

Corollary 3. *Let $h(z)$ be meromorphic function in the half-plane* $\mathrm{Re}z \geq 0$ *(with exception the possible branch point $z = 0$) with poles z_k, $\pm iy_k$ ($\neq \gamma_{\nu,i}$),* $\mathrm{Re}z_k, y_k > 0$. *If this function satisfy condition*

$$h(xe^{\pi i/2}) = -h(xe^{-\pi i/2}) \tag{4.15}$$

and the integral (4.12) exists then

$$\lim_{b\to+\infty} \left\{ \frac{\pi^2}{2} \sum_{k=1}^{m} h(\gamma_{\nu,k}) T_\nu^{AB}(\lambda, \gamma_{\nu,k}) + r_{2\nu}[h(z)] - \text{p.v.} \int_0^b \frac{h(x)dx}{\bar{J}_\nu^2(x) + \bar{Y}_\nu^2(x)} \right\} =$$
$$= -\frac{\pi}{2} \sum_{\eta_k=0, iy_k} (2 - \delta_{0\eta_k}) \, \mathrm{Res}_{z=\eta_k} \left[\frac{\bar{H}_\nu^{(1)}(\lambda z)}{\bar{H}_\nu^{(1)}(z)} \frac{h(z)}{C_\nu^{AB}(\lambda, z)} \right], \tag{4.16}$$

where in the lhs $\gamma_{\nu,m} < b < \gamma_{\nu,m+1}$.

Let us consider a special applications of the formula (4.16) for $A = 1$, $B = 0$. The generalizations of these results for general A, B under the conditions given above are straightforward.

Theorem 4. *Let the function $F(z)$ be meromorphic in the right half-plane* $\mathrm{Re}z \geq 0$ *(with the possible exception $z = 0$) with poles $z_k, \pm iy_k$ ($\neq \gamma_{\nu,i}$), y_k,* $\mathrm{Re}z_k > 0$. *If it satisfy condition*

$$F(xe^{\pi i/2}) = (-1)^{m+1} F(xe^{-\pi i/2}), \tag{4.17}$$

with an integer m, and to the one of two inequalities

$$|F(z)| < \varepsilon(x)e^{a_1|y|} \quad or \quad |F(z)| < M|z|^{-\alpha}e^{a_2|y|}, \quad |z| \to \infty, \tag{4.18}$$

with $a_1 < \min(2\lambda - \sigma - 1, \sigma - 1) \equiv a_2$, $\sigma > 0$, $\varepsilon(x) \to 0$ for $x \to +\infty$, $\alpha > 1$, then

$$\sum_{k=1}^{\infty} \frac{\gamma_{\nu,k}F(\gamma_{\nu,k})}{J_\nu^2(\gamma_{\nu,k})/J_\nu^2(\lambda\gamma_{\nu,k}) - 1} \left[J_\nu(\gamma_{\nu,k})Y_{\nu+m}(\sigma\gamma_{\nu,k}) - Y_\nu(\gamma_{\nu,k})J_{\nu+m}(\sigma\gamma_{\nu,k})\right] =$$
$$= \frac{1}{\pi} \sum_{\eta_k=0, iy_k, z_k} (2 - \delta_{0\eta_k}) \operatorname{Res}_{z=\eta_k} \frac{Y_\nu(\lambda z)J_{\nu+m}(\sigma z) - J_\nu(\lambda z)Y_{\nu+m}(\sigma z)}{J_\nu(z)Y_\nu(\lambda z) - J_\nu(\lambda z)Y_\nu(z)} F(z). \tag{4.19}$$

Proof. As a function $h(z)$ in (4.16) let us choose

$$h(z) = F(z)\left[J_\nu(z)Y_{\nu+m}(\sigma z) - Y_\nu(z)J_{\nu+m}(\sigma z)\right], \tag{4.20}$$

which in virtue of (4.18) satisfy condition (4.4). The condition (4.15) is satisfied as well. Hence $h(z)$ satisfy conditions for Corollary 3. The corresponding integral in (4.16) with $h(z)$ from (4.20) can be calculated by using the formula (7.7) (see below). Putting the value of this integral into (4.16) after some manipulations we receive to (4.19). ■

Remark. The formula (4.19) may be derived also by applying to the contour integral

$$\int_{C_h} \frac{Y_\nu(\lambda z)J_{\nu+m}(\sigma z) - J_\nu(\lambda z)Y_{\nu+m}(\sigma z)}{J_\nu(z)Y_\nu(\lambda z) - J_\nu(\lambda z)Y_\nu(z)} F(z)dz \tag{4.21}$$

the residue theorem, where C_h is a rectangle with vertices $\pm ih$, $b \pm ih$. Here the proof is similar to that for Remark to the Corollary 1. ■

Formula similar to (4.19) can be obtained also for the series of type $\sum_{k=1}^{\infty} G(\gamma_{\nu,k})J_\mu(\gamma_{\nu,k}t)$ by using (4.16).

As a function $F(z)$ in (4.19) one can choose, for example,

- function (3.51) for $\rho = m - 2p$, $\sum_l b_l < a_2$, $m < 2p + \sum_l \operatorname{Re}\mu_l + n/2 + 1$, p - integer;
- function (3.52) for $\rho = m - 2p$, $\sum_l b_l < a_2$, $m < 2p + 2n + 1$;
- function (3.54) for $\rho = m - 2p$, $a_l > 0$, $\operatorname{Re}\mu_l \geq 0$ (for $\operatorname{Re}\mu_l < 0$, $k_l > |\operatorname{Re}\mu_l|$), $\sum_{l=1}^{n} |a_l - b_l| < a_2$, $m < 2p + n - \sum_l |k_l| + 1$.

For $F(z) = 1/z$, $m = 0$ one obtains

$$\sum_{k=1}^{\infty} \frac{J_\nu(\gamma_{\nu,k})Y_\nu(\sigma\gamma_{\nu,k}) - Y_\nu(\gamma_{\nu,k})J_\nu(\sigma\gamma_{\nu,k})}{J_\nu^2(\gamma_{\nu,k})/J_\nu^2(\lambda\gamma_{\nu,k}) - 1} = \frac{\sigma^\nu}{\pi} \frac{(\lambda/\sigma)^{2\nu} - 1}{\lambda^{2\nu} - 1}, \quad \lambda \geq \sigma > 1. \tag{4.22}$$

By similar way it can be seen that

$$\sum_{k=1}^{\infty} \frac{\gamma_{\nu,k}^2\left[J_\nu(\gamma_{\nu,k})Y_\nu(\sigma\gamma_{\nu,k}) - Y_\nu(\gamma_{\nu,k})J_\nu(\sigma\gamma_{\nu,k})\right]}{\left(\gamma_{\nu,k}^2 - c^2\right)\left[J_\nu^2(\gamma_{\nu,k})/J_\nu^2(\lambda\gamma_{\nu,k}) - 1\right]} = \frac{1}{\pi} \frac{Y_\nu(\lambda c)J_\nu(\sigma c) - J_\nu(\lambda c)Y_\nu(\sigma c)}{J_\nu(c)Y_\nu(\lambda c) - J_\nu(\lambda c)Y_\nu(c)}, \tag{4.23}$$

$$\sum_{k=1}^{\infty} \frac{J_\nu(\gamma_{\nu,k})Y_\nu(\sigma\gamma_{\nu,k}) - Y_\nu(\gamma_{\nu,k})J_\nu(\sigma\gamma_{\nu,k})}{J_\nu^2(\gamma_{\nu,k})/J_\nu^2(\lambda\gamma_{\nu,k}) - 1} \prod_{l=1}^{p} \gamma_{\nu,k}^{-\mu_l} J_{\mu_l}(b_l\gamma_{\nu,k}) =$$
$$= \frac{\sigma^\nu}{\pi} \frac{(\lambda/\sigma)^{2\nu} - 1}{\lambda^{2\nu} - 1} \prod_{l=1}^{p} \frac{b_l^{\mu_l}}{2^{\mu_l}\Gamma(\mu_l + 1)}, \quad b \equiv \sum_{1}^{p} b_l < \sigma - 1, \ \operatorname{Re}\mu_l + \frac{p}{2} + 1 > \delta_{b,\sigma-1}, \tag{4.24}$$

where $\mathrm{Re}c \geq 0$, $b_l > 0$, $\lambda \geq \sigma > 1$, $\mu_l \neq -1, -2, \ldots$.

So far in this section we have considered series over zeros of the function $C_\nu^{AB}(\lambda, z)$. The similar results can be obtained also for the series containing zeros of the function

$$C_{1\nu}(\lambda, z) = J_\nu'(z)Y_\nu(\lambda z) - Y_\nu'(z)J_\nu(\lambda z), \tag{4.25}$$

(on properties of zeroes of these function see [20, 7, 21]). The corresponding furmulae for the zeros $\gamma_{1\nu,k}$ of this function can be obtained from those for $C_\nu^{AB}(\lambda, z)$ by replacements

$$\begin{gathered} T_\nu^{AB}(\lambda, z) \to \frac{z}{J_\nu^2(z)/J_\nu^2(\lambda z) - 1 + \nu^2/z^2} \\ \bar{f}(z) \to f'(z), \quad \bar{f}(\lambda z) \to f(\lambda z), \quad f = J_\nu, Y_\nu, H_\nu^{(1,2)}, I_\nu, K_\nu, \quad C_\nu^{AB} \to C_{1\nu}. \end{gathered} \tag{4.26}$$

The physiacl apllications of the formulae derived in this section will be considered below in Sections 9 and 12.

5 Applications to integrals involving Bessel functions

The applications of GAPF to infinite integrals involving some combinations of Bessel functions lead to the interesting results [11, 12]. First of all one can express integrals over Bessel functions through the integrals involving modified functions. Let us substitute in the formula (3.18)

$$f(z) = F(z)\bar{J}_\nu(z). \tag{5.1}$$

For the function $F(z)$ having no poles at $z = \lambda_{\nu,k}$ the sum over zeros of $\bar{J}_\nu(z)$ is zero. The conditions (3.4) and (3.11) may be written in terms of $F(z)$ as

$$|F(z)| < \varepsilon_1(x)e^{c_1|y|} \quad \text{or} \quad |F(z)| < M|z|^{-\alpha_1}e^{|y|}, \quad |z| \to \infty, \tag{5.2}$$

with $c_1 < 1$, $x^{1/2-\delta_{B0}}\varepsilon_1(x) \to 0$ for $x \to \infty$, $\alpha_1 > \alpha_0 = 3/2 - \delta_{B0}$, and

$$F(ze^{\pi i}) = -e^{\nu\pi i}F(z) + o\left(z^{|\mathrm{Re}\nu|-1}\right). \tag{5.3}$$

Hence for the function $F(z)$ satisfying conditions (5.2) and (5.3) it follows from (3.18) that

$$\begin{aligned} \text{p.v.} \int_0^\infty F(x)\bar{J}_\nu(x)dx &= r_{1\nu}\left[F(z)\bar{J}_\nu(z)\right] + \frac{\pi}{2}\mathrm{Res}_{z=0}F(z)\bar{Y}_\nu(z) + \\ &+ \frac{1}{\pi}\int_0^\infty \bar{K}_\nu(x)\left[e^{-\nu\pi i/2}F(xe^{\pi i/2}) + e^{\nu\pi i/2}F(xe^{-\pi i/2})\right]dx. \end{aligned} \tag{5.4}$$

In expression (3.9) for $r_{1\nu}$ the points z_k are poles of the meromorpic function $F(z)$, $\mathrm{Re}z_k > 0$. On the base of Remark after Theorem 3 the formula (5.4) may be generalized for the functions $F(z)$ with purely imaginary poles $\pm iy_k$, $y_k > 0$ and satisfying condition

$$F(ze^{\pi i}) = -e^{\nu\pi i}F(z) + o\left((z \mp iy_k)^{-1}\right), \quad z \to \pm iy_k. \tag{5.5}$$

The corresponding formula is obtained from (5.4) by adding residue terms for $z = iy_k$ in the form of (5.7) (see below) and taking the principal value of the integral on the right.

The same substitution (5.1) with the function $F(z)$ satisfying the conditions (5.2) and

$$F(xe^{\pi i/2}) = -e^{\nu\pi i}F(xe^{-\pi i/2}) \tag{5.6}$$

for real x, into the formula (3.33) yields the following result

$$\text{p.v.} \int_0^\infty F(x)\bar{J}_\nu(x)dx = r_{1\nu}\left[F(z)\bar{J}_\nu(z)\right] + \frac{\pi i}{2}\sum_{\eta_k=0,iy_k}(2-\delta_{0\eta_k})\,\mathrm{Res}_{z=\eta_k}F(z)\bar{H}_\nu^{(1)}(z). \tag{5.7}$$

In (3.9) now summation is over the poles z_k, $\mathrm{Re}z_k > 0$ of the meromorphic function $F(z)$, and $\pm iy_k$, $y_k > 0$ are purely imaginary poles of this function. Recall that the possible real poles of $F(z)$ are such, that integral on the left of (5.7) exists.

For the functions $F(z) = z^{\nu+1}\tilde{F}(z)$, with $\tilde{F}(z)$ being analytic in the right half-plane and even along the imaginary axis, $\tilde{F}(ix) = \tilde{F}(-ix)$, one obtains

$$\int_0^\infty x^{\nu+1}\tilde{F}(x)\bar{J}_\nu(x)dx = 0. \tag{5.8}$$

This result for $B = 0$ (see (3.1)) have been given previously in [24]. The another result of [24] is obtained from (5.7) choosing $F(z) = z^{\nu+1}\tilde{F}(z)/(z^2 - a^2)$.

Formulae similar to (5.4) and (5.7) can be derived for Neumann function $Y_\nu(z)$. Let for the function $F(z)$ the integtral p.v. $\int_0^\infty F(x)\bar{Y}_\nu(x)dx$ exists. Let us substitute in the formula (2.11)

$$f(z) = Y_\nu(z)F(z), \quad g(z) = -iJ_\nu(z)F(z) \tag{5.9}$$

and consider the limit $a \to +0$. The summands containing residues may be presented in the form

$$\begin{aligned} R[f(z), g(z)] &= \pi\sum_k \mathrm{Res}_{\mathrm{Im}z_k>0} H_\nu^{(1)}(z)F(z) + \pi\sum_k \mathrm{Res}_{\mathrm{Im}z_k<0} H_\nu^{(2)}(z)F(z) + \\ &+ \pi\sum_k \mathrm{Res}_{\mathrm{Im}z_k=0} J_\nu(z)F(z) \equiv r_{3\nu}[F(z)], \end{aligned} \tag{5.10}$$

where z_k ($\mathrm{Re}z_k > 0$) are the poles of $F(z)$ in the right half-plane. Now the following results can be prooved by using (2.11):

1) If the meromorphic function $F(z)$ has no poles on the imaginary axis and satisfy the condition (5.2) then

$$\text{p.v.}\int_0^\infty F(x)Y_\nu(x)dx = r_{3\nu}[F(z)] - \frac{i}{\pi}\int_0^\infty K_\nu(x)\left[e^{-\nu\pi i/2}F(xe^{\pi i/2}) - e^{\nu\pi i/2}F(xe^{-\pi i/2})\right]dx \tag{5.11}$$

and

2) If the meromorphic function $F(z)$ satisfy the conditions

$$F(xe^{\pi i/2}) = e^{\nu\pi i}F(xe^{-\pi i/2}) \tag{5.12}$$

and (5.2) then one has

$$\text{p.v.}\int_0^\infty F(x)Y_\nu(x)dx = r_{3\nu}[F(z)] + \pi\sum_k \mathrm{Res}_{z=iy_k} H_\nu^{(1)}(z)F(z), \tag{5.13}$$

where $\pm iy_k$, $y_k > 0$ are purely imaginary poles of $F(z)$.

From (5.13) it directly follows that for $F(z) = z^\nu\tilde{F}(z)$, with $\tilde{F}(z)$ being even along the imaginary axis, $\tilde{F}(ix) = \tilde{F}(-ix)$, and analytic in the right half-plane [24]

$$\text{p.v.}\int_0^\infty x^\nu\tilde{F}(x)Y_\nu(x)dx = 0, \tag{5.14}$$

if the condition (5.2) takes place.

Let us consider more general case. Let the function $F(z)$ satisfy the condition

$$F(ze^{-\pi i}) = -e^{-\lambda\pi i}F(z) \tag{5.15}$$

for $\arg z = \pi/2$. In GAPF as functions $f(z)$ and $g(z)$ we choose

$$\begin{aligned} f(z) &= F(z)\left[J_\nu(z)\cos\delta + Y_\nu(z)\sin\delta\right] \\ g(z) &= -iF(z)\left[J_\nu(z)\sin\delta - Y_\nu(z)\cos\delta, \quad \delta = (\lambda-\nu)\pi/2\right], \end{aligned} \tag{5.16}$$

with $g(z)-(-1)^k f(z) = H_\nu^{(k)}(z)F(z)\exp[(-1)^k i\delta]$, $k=1,2$. It can be seen that for such a choice the integral on rhs of (2.11) for $a \to 0$ is equal to

$$\pi i \sum_{\eta_k = iy_k} \mathrm{Res}_{z=\eta_k} H_\nu^{(1)}(z)F(z)e^{-i\delta}, \tag{5.17}$$

where $\pm iy_k$, $y_k > 0$, as above, are purely imaginary poles of $F(z)$. Substituting (5.16) into (2.11) and using (2.3) we obtain [11, 12]

Corollary 4. *Let $F(z)$ be meromorphic function for* $\mathrm{Re}z \geq 0$ *(except possibly at $z=0$) with poles z_k, $\pm iy_k$; $y_k, \mathrm{Re}z_k > 0$. If this function satisfies conditions (5.2) (for $B=0$) and (5.15) then*

$$\begin{aligned} \mathrm{p.v.}\int_0^\infty F(x)\left[J_\nu(x)\cos\delta + Y_\nu(x)\sin\delta\right]dx = \pi i \Bigg\{ \sum_{z_k} \mathrm{Res}_{\mathrm{Im}z_k>0} H_\nu^{(1)}(z)F(z)e^{-i\delta} - \\ - \sum_{z_k} \mathrm{Res}_{\mathrm{Im}z_k<0} H_\nu^{(2)}(z)F(z)e^{i\delta} - i\sum_{z_k} \mathrm{Res}_{\mathrm{Im}z_k=0}\left[J_\nu(z)\sin\delta - Y_\nu(z)\cos\delta\right]F(z) + \\ + \sum_{\eta_k=iy_k} \mathrm{Res}_{z=\eta_k} H_\nu^{(1)}(z)F(z)e^{-i\delta} \Bigg\}, \end{aligned} \tag{5.18}$$

where it is assumed that integral on the left exists.

In particular for $\delta = \pi n$, $n = 0, 1, 2...$ the formula (5.7) follows from here in the case $B = 0$.

One will find a great many particular cases of the formulae (5.7) and (5.18) looking at the standard books and tables of known integrals with Bessel functions (see, e.g., [7, 20, 22, 23, 25, 26, 27, 28, 29]). Some special examples are given in the next section.

6 Integrals involving Bessel functions: Illustrations of general formulae

To illustrate the applications of the general formulae from previous section first of all consider integrals involving the function $\bar{J}_\nu(z)$. Let us introduce the functional

$$A_{\nu m}[G(z)] \equiv \mathrm{p.v.}\int_0^\infty z^{\nu-2m-1}G(z)\bar{J}_\nu(z)dz \tag{6.1}$$

with m being an integer. Let $F_1(z)$ be an analytic function in the right half-plane satisfying condition

$$F_1(xe^{\pi i/2}) = F_1(xe^{-\pi i/2}), \quad F_1(0) \neq 0 \tag{6.2}$$

(the case when $F_1(z) \sim z^q$, $z \to 0$ with an integer q can be reduced to this one by redefinitions of $F_1(z)$ and m). From (5.7) the following results can be obtained [11, 12]

$$\begin{aligned}
A_{\nu m}[F_1(z)] &= A_{\nu m}^{(0)}[F_1(z)] \equiv -\frac{\pi(1+\mathrm{sgn}m)}{4(2m)!}\left(\frac{d}{dz}\right)^{2m}\left[z^\nu \bar{Y}_\nu(z)F_1(z)\right]|_{z=0} \qquad (6.3)\\
A_{\nu m}\left[\frac{F_1(z)}{z^2-a^2}\right] &= -\frac{\pi}{2}a^{\nu-2m-2}\bar{Y}_\nu(a)F_1(a) + A_{\nu m}^{(0)}\left[\frac{F_1(z)}{z^2-a^2}\right], \qquad (6.4)\\
A_{\nu m}\left[\frac{F_1(z)}{z^4-a^4}\right] &= -\frac{a^{\nu-2m-4}}{2}\left[\frac{\pi}{2}\bar{Y}_\nu(a)F_1(a) - (-1)^m \bar{K}_\nu(a)F_1(ia)\right] + \\
&\quad + A_{\nu m}^{(0)}\left[\frac{F_1(z)}{z^4-a^4}\right], \qquad (6.5)\\
A_{\nu m}\left[\frac{F_1(z)}{(z^2-c^2)^{p+1}}\right] &= \frac{\pi i}{2^{p+1}p!}\left(\frac{d}{cdc}\right)^p\left[c^{\nu-2m-2}F_1(c)H_\nu^{(1)}(c)\right] + A_{\nu m}^{(0)}\left[\frac{F_1(z)}{(z^2-c^2)^{p+1}}\right] \qquad (6.6)\\
A_{\nu m}\left[\frac{F_1(z)}{(z^2+a^2)^{p+1}}\right] &= \frac{(-1)^{m+p+1}}{2^p\cdot p!}\left(\frac{d}{ada}\right)^p\left[a^{\nu-2m-2}K_\nu(a)F_1(ae^{\pi i/2})\right] + \\
&\quad + A_{\nu m}^{(0)}\left[\frac{F_1(z)}{(z^2+a^2)^{p+1}}\right], \qquad (6.7)
\end{aligned}$$

and etc.(note that $A_{\nu m}^{(0)} = 0$ for $m < 0$). Here $a > 0$, $0 < \arg c < \pi/2$, and we have assumed that $\mathrm{Re}\nu > 0$. To secure convergence at the origin the condition $\mathrm{Re}\nu > m$ is necessary. In the last two formulae we have used the identity

$$\left(\frac{d}{dz}\right)^p\left[\frac{zF(z)}{(z+b)^{p+1}}\right]_{z=b} = \frac{1}{2^{p+1}}\left(\frac{d}{bdb}\right)^p F(b). \qquad (6.8)$$

Note that (6.7) can be obtained also from (6.6) in the limit $\mathrm{Re}c \to 0$. For the case $F_1 = 1$, $m = -1$ of (6.7) see, for example, [20]. In (6.3)-(6.7) as a function $F_1(z)$ we can choose:

- function (3.51) for $\rho = 1$, $\mathrm{Re}\nu < \sum \mathrm{Re}\mu_l + 2m + (n+1)/2 - \delta_{b1} + \delta_{B0}$, $b = \sum b_l \le 1$, $b_l > 0$;
- function (3.52) with $\rho = 1$, $\mathrm{Re}\nu < 2(m+n) + 1/2 - \delta_{b1} + \delta_{B0}$, $b = \sum b_l \le 1$;
- function (3.53) for $\rho = 1$, $\mathrm{Re}\nu < 2m - \sum \mathrm{Re}\mu_l + (n+1)/2 - \delta_{b1} + \delta_{B0}$, $\mu_l > 0$ is half of an odd integer, $b = \sum b_l \le 1$;
- function (3.54) for $\mathrm{Re}\nu < 2m + n - \sum |k_l| + 1/2 + \delta_{B0} - \delta_{\tilde{a}1}$, $\tilde{a} = \sum |a_l - b_l| \le 1$, $a_l \ge 0$, k_l - integer.

Here we have written the conditions for (6.3). The corresponding ones for (6.4), (6.5), (6.6),(6.7) are obtained from these by adding on the rhs of inequalities for $\mathrm{Re}\nu$, respectively 2, 4, $2(p+1)$, $2(p+1)$. In (6.3)-(6.7) we can choose also any combinations of the functions (3.51)-(3.54) with appropriate conditions.

For concrete evaluations of $A_{\nu m}^{(0)}$ in special cases it is useful the following formula

$$\lim_{z\to 0}\left(\frac{d}{dz}\right)^{2m} f_1(z) = (2m-1)!!\lim_{z\to 0}\left(\frac{d}{zdz}\right)^m f_1(z), \qquad (6.9)$$

valid for the function $f_1(z)$ satisfying condition $f_1(-z) = f_1(z) + o(z^{2m})$, $z \to 0$. From here, for instance, it follows that for $z \to 0$

$$\left(\frac{d}{dz}\right)^{2m}\left[z^\nu Y_\nu(bz)F_1(z)\right] = -(2m-1)!!\frac{2^{\nu-m}}{\pi b^{\nu-m}}\sum_{k=0}^{m}\binom{m}{k}2^k\frac{\Gamma(\nu-m+k)}{b^{2k}}\left(\frac{d}{zdz}\right)^k F_1(z), \qquad (6.10)$$

where we have used the standard formula for the derivative $(d/zdz)^n$ of cylinder functions (see [21]). From (6.3) one obtains ($B=0$)

$$\int_0^\infty z^{\nu-2m-1}J_\nu(z)\prod_{l=1}^n\left(z^2+z_l^2\right)^{-\mu_l/2}J_{\mu_l}(b_l\sqrt{z^2+z_l^2})dz =$$
$$= -\frac{\pi}{2^{m+1}m!}\left(\frac{d}{zdz}\right)^m\left[z^\nu Y_\nu(z)\prod_{l=1}^n\left(z^2+z_l^2\right)^{-\mu_l/2}J_{\mu_l}(b_l\sqrt{z^2+z_l^2})\right]_{z=0}, \qquad (6.11)$$

for $m\geq 0$ and the integral is zero for $m<0$. Here $\mathrm{Re}\nu>0$, $b\equiv\sum_1^n b_l\leq 1$, $b_l>0$, $m<\mathrm{Re}\nu<\sum_{l=1}^n\mathrm{Re}\mu_l+2m+(n+3)/2-\delta_{b1}$. In particular case $m=0$ the Gegenbauer integral follows from here [20, 7]. In the limit $z_l\to 0$ from (6.11) the value of integral $\int_0^\infty z^{\nu-2m-1}J_\nu(z)\prod_{l=1}^n z^{-\mu_l}J_{\mu_l}(z)dz$ can be obtained.

By using (5.18) the formulae similar to (6.3)-(6.7) may be derived for the integrals of type

$$B_\nu[G(z)]\equiv \mathrm{p.v.}\int_0^\infty G(z)\left[J_\nu(z)\cos\delta+Y_\nu(z)\sin\delta\right]dz, \quad \delta=(\lambda-\nu)\pi/2. \qquad (6.12)$$

It directly follows from Corollary 4 that for function $F(z)$ analytic for $\mathrm{Re}z\geq 0$ and satisfying conditions (5.2) and (5.15) the following formulae take place

$$B_\nu[F(z)] = 0 \qquad (6.13)$$
$$B_\nu\left[\frac{F(z)}{z^2-a^2}\right] = \pi F(a)\left[J_\nu(a)\sin\delta-Y_\nu(a)\cos\delta\right]/2, \qquad (6.14)$$
$$B_\nu\left[\frac{F(z)}{z^4-a^4}\right] = \frac{\pi}{4a^3}F(a)\left[J_\nu(a)\sin\delta-Y_\nu(a)\cos\delta\right]+\frac{i}{2a^3}K_\nu(a)F(ia)e^{-i\lambda\pi/2}, \qquad (6.15)$$
$$B_\nu\left[\frac{F_1(z)}{(z^2-c^2)^{p+1}}\right] = \frac{\pi i}{2^{p+1}\cdot p!}\left(\frac{d}{cdc}\right)^p\left[c^{-1}F(c)H_\nu^{(1)}(c)\right]e^{-i\delta} \qquad (6.16)$$
$$B_\nu\left[\frac{F_1(z)}{(z^2+a^2)^{p+1}}\right] = \frac{(-1)^{p+1}}{2^p\cdot p!}\left(\frac{d}{ada}\right)^p\left[a^{-1}F(ae^{\pi i/2})K_\nu(a)\right]e^{-i\pi\lambda/2}, \qquad (6.17)$$

where $a>0$, $0<\arg c\leq\pi/2$. To obtain the last two formulae we have used the identity (6.8). The formula (6.13) generalizes the result of [24] (the cases $\lambda=\nu$ and $\lambda=\nu+1$). From the last formula taking $F(z)=z^{\lambda-1}$ we obtain result given in [20]. In (6.13) - (6.17) as a function $F(z)$ one can choose (the constraints on parameters are written for the formula (6.13); the corresponding constraints for (6.14), (6.15), (6.16), (6.17) are obtained from given ones by adding the summands 2, 4, $2(p+1)$, $2(p+1)$ to the rhs of inequalities, correspondingly):

- function (3.51) for $\rho=\lambda$, $|\mathrm{Re}\nu|<\mathrm{Re}\rho<\sum\mathrm{Re}\mu_l+(n+3)/2-\delta_{b1}$, $b=\sum_l b_l\leq 1$;
- function (3.52) for $\rho=\lambda$, $|\mathrm{Re}\nu|<\mathrm{Re}\rho<3/2-\delta_{b1}$, $b=\sum_l b_l\leq 1$;
- function

$$z^{\rho-1}\prod_{l=1}^n\left[J_{\mu_l+k_l}(a_lz)Y_{\mu_l}(b_lz)-Y_{\mu_l+k_l}(a_lz)J_{\mu_l}(b_lz)\right], \quad \lambda=\rho+\sum_{l=1}^n k_l,\ a_l>0, \qquad (6.18)$$
$$|\mathrm{Re}\nu|+\sum|k_l|<\mathrm{Re}\rho<n+3/2-\delta_{c1},\ c=\sum|a_l-b_l|\leq 1,\ \mathrm{Re}\mu_l\geq 0$$

  (for $\mathrm{Re}\mu_l<0$ one has $k_l>|\mathrm{Re}\mu_l|$).

Any combination of these functions with appropriate conditions on parameters can be choosed as well.

Now consider integrals which can be expressed via series by using (5.7) and (5.18). In (5.7) let us choose the function

$$F(z) = \frac{z^{\nu-2m}F_1(z)}{\sinh \pi z}, \tag{6.19}$$

where $F_1(z)$ is the same as in the formulae (6.3) - (6.6). As the points $\pm i, \pm 2i, \ldots$ are simple poles for $F(z)$ from (5.7) one obtains

$$\int_0^\infty \frac{z^{\nu-2m}}{\sinh(\pi z)} F_1(z)\bar{J}_\nu(z)dz = A^{(0)}_{\nu m}\left[\frac{zF_1(z)}{\sinh(\pi z)}\right] + \frac{2}{\pi}\sum_{k=1}^{\infty}(-1)^{m+k}k^{\nu-2m}\bar{K}_\nu(k)F_1(ik), \tag{6.20}$$

where $A^{(0)}_{\nu m}[f(z)]$ is defined by (6.3) and $\mathrm{Re}\nu > m$. The corresponding costraints on $F_1(z)$ follow directly from (5.2). The particular case of this formula when $F_1(z) = \sinh(az)/z$ and $m = -1$ is given in [20]. As a function $F_1(z)$ here one can choose any of functions (3.51)-(3.54) with $\rho = 1$ and $\tilde{a}$, $\sum_l b_l < 1$. From (6.20) it follows that

$$\int_0^\infty \frac{z^{\nu-2m}}{\sinh(\pi z)} J_\nu(z)\prod_{l=1}^{n} z^{-\mu_l}I_{\mu_l}(b_l z)dz = A^{(0)}_{\nu m}\left[\frac{z}{\sinh(\pi z)}\prod_{l=1}^{n} z^{-\mu_l}I_{\mu_l}(b_l z)\right] +$$
$$+\frac{2}{\pi}\sum_{k=1}^{\infty}(-1)^{m+k}K_\nu(k)\prod_{l=1}^{n}k^{-\mu_l}J_{\mu_l}(b_l k), \quad b_l > 0,\ \pi - \sum_{l=1}^{n} b_l > 0,\ \mathrm{Re}\nu > m. \tag{6.21}$$

In similar way from (5.18) it can be derived the following formula

$$\int_0^\infty \frac{zF(z)}{\sinh(\pi z)}\left[J_\nu(z)\cos\delta + Y_\nu(z)\sin\delta\right]dz = \frac{2i}{\pi}e^{-i\lambda\pi/2}\sum_{k=1}^{\infty}(-1)^k kK_\nu(k)F(ik). \tag{6.22}$$

Constraints on the function $F(z)$ immediately follow from Corollary 4. Instead of this function we can choose the functions (3.51), (3.52), (3.54).

As it have been mentioned above adding residue terms $\pi i \mathrm{Res}_{z=iy_k}F(z)\bar{H}^{(1)}_\nu(z)$ to the rhs of (5.4) this formula may be generalized for the functions having purely imaginary poles $\pm iy_k$, $y_k > 0$, provided the condition (5.5) is satisfied. As an application let us choose

$$F(z) = \frac{z^\nu F_1(z)}{e^{2\pi z/b} - 1}, \quad F_1(-z) = F_1(z), \quad b > 0 \tag{6.23}$$

with an analytic function $F_1(z)$. The function (6.23) satisfy condition (5.5) and have poles $\pm ikb$, $k = 0, 1, 2 \ldots$. The additional constraint directly follows from (5.2). Then one obtains

$$\int_0^\infty \frac{x^\nu J_\nu(x)}{e^{2\pi x/b} - 1}F_1(x)dx = \frac{2}{\pi}\sum_{k=0}^{\infty}{}'(bk)^\nu K_\nu(bk)F_1(ibk) - \frac{1}{\pi}\int_0^\infty x^\nu K_\nu(x)F_1(ix)dx, \tag{6.24}$$

where the prime indicates that the $m = 0$ term is to be halved. For the particular case $F_1(z) = 1$, using the relation

$$\sum_{k=0}^{\infty}{}'(bk)^\nu K_\nu(bk) = \frac{\sqrt{\pi}}{b}2^\nu\Gamma(\nu + 1/2)\sum_{n=0}^{\infty}{}'\left[\left(\frac{2\pi n}{b}\right)^2 + 1\right]^{-\nu-1/2} \tag{6.25}$$

and the known value for the integral on the right, we immediately obtain the result given in [20]. The relation (6.25) can be proved by using the formulae

$$K_\nu(z) = \frac{2^\nu\Gamma(\nu + 1/2)}{\sqrt{\pi}z^\nu}\int_0^\infty \frac{\cos zt\, dt}{(t^2 + 1)^{\nu+1/2}}, \qquad \sum_{k=-\infty}^{+\infty} e^{ikz} = 2\pi\sum_{n=-\infty}^{+\infty}\delta(z - 2\pi n), \tag{6.26}$$

(for the integral representation of Macdonald's function see [20]), $\delta(z)$ is the Dirac delta function.

7 Formulae for integrals involving $J_\nu(z)Y_\mu(\lambda z) - Y_\nu(z)J_\mu(\lambda z)$

In this section we shall consider the applications of GAPF to the integrals involving the function $J_\nu(z)Y_\mu(\lambda z) - Y_\nu(z)J_\mu(\lambda z)$. In the formula (2.11) we substitute

$$f(z) = -\frac{1}{2i}F(z)\sum_{l=1}^{2}(-1)^l\frac{H_\mu^{(l)}(\lambda z)}{H_\nu^{(l)}(z)}, \quad g(z) = \frac{1}{2i}F(z)\sum_{l=1}^{2}\frac{H_\mu^{(l)}(\lambda z)}{H_\nu^{(l)}(z)}. \tag{7.1}$$

For definiteness we consider the case $\lambda > 1$ (for $\lambda < 1$ the expression for $g(z)$ have to be choosen with opposite sign). The conditions (2.1) and (2.10) are satisfied if the function $F(z)$ is constrained by the one of the following two inequalities

$$|F(z)| < \varepsilon(x)e^{c|y|}, \quad c < \lambda - 1,\, \varepsilon(x) \to 0,\, x \to +\infty \tag{7.2}$$

or

$$|F(z)| < M|z|^{-\alpha}e^{(\lambda-1)|y|}, \quad \alpha > 1,\, |z| \to \infty,\, z = x + iy. \tag{7.3}$$

Then from (2.11) it follows that for the function $F(z)$ meromorphic in $\mathrm{Re}z \geq a > 0$ one has

$$\begin{aligned}\text{p.v.}\int_0^\infty \frac{J_\nu(x)Y_\mu(\lambda x) - J_\mu(\lambda x)Y_\nu(x)}{J_\nu^2(x) + Y_\nu^2(x)}F(x)dx = r_{1\mu\nu}[F(z)] + \\ +\frac{1}{2i}\left[\int_a^{a+i\infty}F(z)\frac{H_\mu^{(1)}(\lambda z)}{H_\nu^{(1)}(z)}dz - \int_a^{a-i\infty}F(z)\frac{H_\mu^{(2)}(\lambda z)}{H_\nu^{(2)}(z)}dz\right],\end{aligned} \tag{7.4}$$

where we have introduced the notation

$$r_{1\mu\nu}[F(z)] = \frac{\pi}{2}\sum_k \mathrm{Res}_{\mathrm{Im}z_k=0}\left[F(z)\sum_{l=1}^{2}\frac{H_\mu^{(l)}(\lambda z)}{H_\nu^{(l)}(z)}\right] + \pi\sum_k\sum_{l=1}^{2}\mathrm{Res}_{(-1)^l\mathrm{Im}z_k<0}\left[F(z)\frac{H_\mu^{(l)}(\lambda z)}{H_\nu^{(l)}(z)}\right]. \tag{7.5}$$

The most important case for the applications is the limit $a \to 0$. The following statements take place [11, 12]:

Theorem 5. *Let the function $F(z)$ be meromorphic for $\mathrm{Re}z \geq 0$ (except the possible branch point $z = 0$) with poles z_k, $\pm iy_k$ ($y_k, \mathrm{Re}z_k > 0$). If this function satisfy conditions (7.2) or (7.3) and*

$$F(xe^{\pi i/2}) = -e^{(\mu-\nu)\pi i}F(xe^{-\pi i/2}), \tag{7.6}$$

then for values of ν for which the function $H_\nu^{(1)}(z)$ ($H_\nu^{(2)}(z)$) have no zeros for $0 \leq argz \leq \pi/2$ ($-\pi/2 \leq argz \leq 0$) the following formula is valid

$$\begin{aligned}\text{p.v.}\int_0^\infty \frac{J_\nu(x)Y_\mu(\lambda x) - Y_\nu(x)J_\mu(\lambda x)}{J_\nu^2(x) + Y_\nu^2(x)}F(x)dx = r_{1\mu\nu}[F(z)] + \\ +\frac{\pi}{2}\sum_{\eta_k=0,iy_k}(2-\delta_{0\eta_k})\,\mathrm{Res}_{z=\eta_k}F(z)\frac{H_\mu^{(1)}(\lambda z)}{H_\nu^{(1)}(z)},\end{aligned} \tag{7.7}$$

where it is assumed that the integral on the left exists.

Proof. From the condition (7.6) it follows that for $\mathrm{arg}z = \pi/2$

$$\frac{H_\mu^{(1)}(\lambda z)}{H_\nu^{(1)}(z)}F(z) = \frac{H_\mu^{(2)}(\lambda z_1)}{H_\nu^{(2)}(z_1)}F(z_1),\ z_1 = e^{-\pi i}, \tag{7.8}$$

and that the possible purely imaginary poles of $F(z)$ are conjugate: $\pm iy_k$, $y_k > 0$. Hence in rhs of (7.4) in the limit $a \to 0$ the term in the square brackets may be presented in the form (it can be seen similarly to (3.32))

$$\left(\int_{\gamma_\rho^+} + \sum_k \int_{C_\rho(iy_k)}\right) \frac{H_\mu^{(1)}(\lambda z)}{H_\nu^{(1)}(z)} F(z)dz + \left(\int_{\gamma_\rho^-} + \sum_k \int_{C_\rho(-iy_k)}\right) \frac{H_\mu^{(2)}(\lambda z)}{H_\nu^{(2)}(z)} F(z)dz \tag{7.9}$$

with the same notations as in (3.32). By using (7.8) and the condition that the integral converges at the origin we obtain

$$\int_{\Omega_\rho^+(\eta_k)} \frac{H_\mu^{(1)}(\lambda z)}{H_\nu^{(1)}(z)} F(z)dz + \int_{\Omega_\rho^-(\eta_k)} \frac{H_\mu^{(2)}(\lambda z)}{H_\nu^{(2)}(z)} F(z)dz = (2 - \delta_{0\eta_k})\,\pi i \,\mathrm{Res}_{z=\eta_k} \frac{H_\mu^{(1)}(\lambda z)}{H_\nu^{(1)}(z)} F(z), \tag{7.10}$$

where $\Omega_\rho^\pm(0) = \gamma_\rho^\pm$, $\Omega_\rho^\pm(iy_k) = C_\rho(\pm iy_k)$. By using this relation from (7.4) we receive the formula (7.7). ■

Note that one can write the residue at $z = 0$ in the form

$$\mathrm{Res}_{z=0} \frac{H_\mu^{(1)}(\lambda z)}{H_\nu^{(1)}(z)} F(z) = \mathrm{Res}_{z=0} \frac{J_\nu(z)J_\mu(\lambda z) + Y_\nu(z)Y_\mu(\lambda z)}{J_\nu^2(z) + Y_\nu^2(z)} F(z) \tag{7.11}$$

as well. Integrals of type (7.7) we have been able to find in literature (see, e.g., [7, 23, 25, 27]) are special cases of this formula. For example, taking $F(z) = J_\nu(z)Y_{\nu+1}(\lambda' z) - Y_\nu(z)J_{\nu+1}(\lambda' z)$ for the integral on the left in (7.7) we obtain $-\lambda^{-\nu}\lambda'^{-\nu-1}$ for $\lambda' < \lambda$ and $\lambda^\nu \lambda'^{-\nu-1} - \lambda^{-\nu}\lambda'^{-\nu-1}$ for $\lambda' > \lambda$ [25]. By taking $z^{2m+1}/(z^2 + a^2)$, $z^{2m+1}/(z^2 - c^2)$ as $F_1(z)$ for $\mu = \nu$ and integer $m \geq 0$ one receive

$$\int_0^\infty \frac{J_\nu(x)Y_\nu(\lambda x) - Y_\nu(x)J_\nu(\lambda x)}{J_\nu^2(x) + Y_\nu^2(x)} \frac{x^{2m+1}}{x^2 + a^2} dx = (-1)^m a^{2m} \frac{\pi}{2} \frac{K_\nu(\lambda a)}{K_\nu(a)}, \ \mathrm{Re}a > 0 \tag{7.12}$$

$$\mathrm{p.v.} \int_0^\infty \frac{J_\nu(x)Y_\nu(\lambda x) - Y_\nu(x)J_\nu(\lambda x)}{J_\nu^2(x) + Y_\nu^2(x)} \frac{x^{2m+1}}{x^2 - c^2} dx = \frac{\pi}{2} c^{2m} \frac{J_\nu(c)J_\nu(\lambda c) + Y_\nu(c)Y_\nu(\lambda c)}{J_\nu^2(c) + Y_\nu^2(c)} \tag{7.13}$$

where $c > 0$, $\lambda > 1$. The particular cases of this formula for $\nu = m = 0$ are given in [25]. In (7.12) taking the limit $a \to 0$ and choosing $m = 0$ we obtain the integral of this type given in [23]. In (7.7) as a function $F(z)$ we can choose (3.51), (3.52), (3.54) (the corresponding conditions for paremeters directly follow from (7.2) or (7.3)) with $\rho = \mu - \nu - 2m$ (m - integer), as well as any products between them and with $\prod_{l=1}^n (z^2 - c_l^2)^{-k_l}$. For instance,

$$\int_0^\infty \frac{J_\nu(x)Y_\nu(\lambda x) - Y_\nu(x)J_\nu(\lambda x)}{J_\nu^2(x) + Y_\nu^2(x)} \prod_{l=1}^n \frac{J_{\mu_l}(b_l\sqrt{x^2 + z_l^2})}{(x^2 + z_l^2)^{\mu_l/2}} \frac{dx}{x} = \frac{\pi}{2\lambda^\nu} \prod_{l=1}^n z^{-\mu_l} J_{\mu_l}(b_l z_l), \tag{7.14}$$

$$b_l,\ \mathrm{Re}\nu > 0,\ \mathrm{Re}z_l \geq 0,\ \lambda > 1,\ \sum_{l=1}^n \mathrm{Re}\mu_l + n/2 + 1 > \delta_{b,\lambda-1},\ b \equiv \sum_{l=1}^n b_l \leq \lambda - 1$$

As another consequence of (7.4) one has:

Theorem 6. *Let $F(z)$ be meromorphic in the right half-plane (with possible exception $z = 0$) with poles z_k, $\mathrm{Re}z_k > 0$, and satisfy conditions (7.2) or (7.3) and*

$$F(ze^{\pi i}) = -e^{(\mu-\nu)\pi i} F(z) + o\left(z^{|\mathrm{Re}\mu| - |\mathrm{Re}\nu| - 1}\right), \quad z \to 0, \tag{7.15}$$

then for values of ν for which the function $H_\nu^{(1)}(z)$ $(H_\nu^{(2)}(z))$ have no zeros for $0 \le argz \le \pi/2$ $(-\pi/2 \le argz \le 0)$ the following formula takes place

$$\text{p.v.} \int_0^\infty \frac{J_\nu(x)Y_\mu(\lambda x) - Y_\nu(x)J_\mu(\lambda x)}{J_\nu^2(x)+Y_\nu^2(x)} F(x)dx = r_{1\mu\nu}[F(z)] + \frac{\pi}{2}\text{Res}_{z=0}\frac{H_\mu^{(1)}(\lambda z)}{H_\nu^{(1)}(z)}F(z) +$$
$$+\frac{1}{2}\int_0^\infty \frac{K_\mu(\lambda x)}{K_\nu(x)}\left[e^{(\nu-\mu)\pi i/2}F(xe^{\pi i/2}) + e^{(\mu-\nu)\pi i/2}F(xe^{-\pi i/2})\right]dx, \quad \lambda > 1, \tag{7.16}$$

provided the integral on the left exists.

Proof. This result immediately follows from (7.4) in the limit $a \to 0$ and from (7.10) with $\eta_k = 0$. ■

For example, by using (7.16) one obtains

$$\int_0^\infty \frac{J_\nu(x)Y_\mu(\lambda x) - Y_\nu(x)J_\mu(\lambda x)}{J_\nu^2(x)+Y_\nu^2(x)} \prod_{l=1}^n J_{\mu_l}(b_l x)dx = \cos\mu_s \int_0^\infty \frac{K_\mu(\lambda x)}{K_\nu(x)}\prod_{l=1}^n I_{\mu_l}(b_l x)dx, \tag{7.17}$$
$$\sum_{l=1}^n \text{Re}\mu_l + |\text{Re}\nu| > |\text{Re}\mu| - 1,\ b = \sum_{l=1}^n b_l \le \lambda - 1,\ b_l > 0,\ n > \delta_{b,\lambda-1},\ \mu_s \equiv \nu - \mu + \sum_{l=1}^n \mu_l.$$

Such relations are convenient in numerical calculations of integrals on the left as the subintegrand on the right at infinity goes to zero exponentially fast.

We have considered the formulas containing $J_\nu(z)Y_\mu(\lambda z) - Y_\nu(z)J_\mu(\lambda z)$. The similar results can be obtained for integrals containing the functions $J'_\nu(z)Y_\mu(\lambda z) - Y'_\nu(z)J_\mu(\lambda z)$ and $J'_\nu(z)Y'_\mu(\lambda z) - Y'_\nu(z)J'_\mu(\lambda z)$.

8 Applications to the Casimir effect. Vacuum energy density and stress inside a perfectly conducting spherical shell

In this and next sections we shall consider applications of the summations formulae obtained in previous sections to the physical problem, namely the Casimir effect. In what follows on the example of spherical and cylindrical geometries we will show that the using of GAPF allows to obtain the regularized values of physical quantities in cases then the explicit dependence of eigenmodes on quantum numbers is complicated and irregular.

Historically the investigation of the Casimir effect for a perfectly conducting spherical shell was motivated by Casimir semiclassical model of an electron. In this model Casimir suggested that Poincare stress to stabilize the charged particle could arise from vacuum quantum fluctuations and the fine structure constant can be determined by a balance between the Casimir force (assumed attractive) and the Coulomb repulsion. However, as it have been shown by Boyer [30], the Casimir energy for the sphere is positive, implying a repulsive force. This result has later been reconsidered by a number of authors [31, 32, 33]. More recently new methods have been developed for this problem including a direct mode summation [38, 39] and zeta function [40, 41, 42] approaches. However the main part of studies have focused on global quantities such as total energy. The investigation of the energy distribution inside a perfectly reflecting spherical shell was made in [34] in the case of QED and in [35] for QCD. The distribution of the other components for the electromagnetic EMT inside as well as outside the shell can be obtained from the results of [36, 37]. In these papers the consideration was carried out in terms of Schwinger's source theory. In [13, 14, 15] the calculations of the regularized values for vev of the EMT components inside and outside the perfectly conducting spherical shell are based

on the generalized Abel-Plana summation formula. Our consideration below is based on this approach.

The main quantities we will consider here are vacuum expectation values (vev) of the energy-momentum tensor (EMT) for the electromagnetic field inside a perfectly conducting spherical shell of radius a. It may be obtained by using the standard formula of mode summation [2, 43]

$$\langle 0|T_{ik}|0\rangle = \sum_{\alpha} T_{ik}(x)\left\{\Psi_{\alpha}^{(-)}(x), \Psi_{\alpha}^{(+)}(x)\right\}, \tag{8.1}$$

where bilinear form $T_{ik}\{f,g\}$ for a field Ψ is given by the classical EMT. Here $|0\rangle$ is the amplitude of the vacuum state, $\left\{\Psi_{\alpha}^{(\pm)}(x)\right\}$ is a complete set of the positive and negative frequency solutions to the field equations, satisfying the boundary conditions, and subscript α may contain discrete and continous components.

In the case of the electromagnetic field inside the perfectly conducting sphere, by using Coulomb gauge for vector potential $\mathbf{A}$, the corresponding system of solutions, regular at $r=0$, can be presented in the form

$$\mathbf{A}_{\alpha} = \omega^{-1}\beta_l(a,\omega)\left\{\begin{array}{lll} j_l(\omega r)\mathbf{X}_{lm}e^{-i\omega t} & \text{if} & \lambda = 0 \\ \omega^{-1}\nabla\times\left[j_l(\omega r)\mathbf{X}_{lm}\right]e^{-i\omega t} & \text{if} & \lambda = 1 \end{array}\right., \quad \alpha = (\omega l m \lambda), \tag{8.2}$$

where $\lambda = 0$ and 1 correspond to the spherical waves of electric and magnetic type (TM and TE - modes). They describe photon with definite values of total momentum l, its projection m, energy ω and parity $(-1)^{l+\lambda+1}$ (units $\hbar = c = 1$ are used). Here the vector spherical harmonics have the form

$$\mathbf{X}_{lm}(\theta,\varphi) = -i\frac{\mathbf{r}\times\nabla}{\sqrt{l(l+1)}}Y_{lm}(\theta,\varphi), \quad l\neq 0, \tag{8.3}$$

with Y_{lm} being spherical function, and $j_l(x) = \sqrt{\pi/2x}J_{l+1/2}(x)$ is spherical Bessel function. The coefficients $\beta_l(a,\omega)$ are determined by the normalization condition

$$\int dV\,\mathbf{A}_{\alpha}\cdot\mathbf{A}_{\alpha'}^{*} = \frac{2\pi}{\omega}\delta_{\alpha\alpha'}, \tag{8.4}$$

where the integration goes over the region inside the sphere. Using the standard formulae for vector spherical harmonics and spherical Bessel functions (see, for example, [44]) one finds

$$\beta_l^2(a,\omega) = 8\omega^3 T_{\nu}(\omega a)/a, \quad \nu = l + 1/2, \tag{8.5}$$

where $T_{\nu}(z)$ is defined in (3.8).

Inside the perfectly conducting sphere the photon energy levels are quantized by standard boundary conditions:

$$\mathbf{r}\times\mathbf{E} = 0, \quad \mathbf{r}\cdot\mathbf{B} = 0, \quad r = a, \tag{8.6}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields. They lead to the following eigenvalue equations with respect to ω

$$j_l(\omega r)|_{r=a} = 0 \quad \text{if } \lambda = 0 \tag{8.7}$$

$$\frac{d}{dr}\left[r j_l(\omega r)\right]_{r=a} = 0 \quad \text{if } \lambda = 1 \tag{8.8}$$

It is well known that these equations have infinite number of real simple roots [20, 21].

By substituting the eigenfunctions into (8.1) with the standard expression of the electromagnetic EMT and after the summation over m by using the standard formulae for vector spherical harmonics (see, for example, [44]) one obtains

$$\langle 0|T_{ik}|0\rangle = \mathrm{diag}\left(\varepsilon,\, -p,\, -p_{\perp},\, -p_{\perp}\right) \tag{8.9}$$

(here index values 1,2,3 correspond to the spherical coordinates r, θ, φ with origin at the sphere centre). Energy density, ε, pressures in transverse, $p_\perp$, and radial, p, directions are determined by relations

$$\begin{aligned} q(a,r) &= \frac{1}{4\pi^2 a}\sum_{\omega l\lambda}\omega^3 T_\nu(\omega a) D_l^{(q)}(\omega r), \quad q=\varepsilon,\, p,\, p_\perp, && (8.10) \\ p(a,r) &= \varepsilon - 2p_\perp, && (8.11) \end{aligned}$$

where the following notations are introduced

$$D_l^{(q)}(y) = \begin{cases} l j_{l+1}^2(y) + (l+1) j_{l-1}^2(y) + (2l+1) j_l^2(y), & q=\varepsilon \\ l(l+1)(2l+1) j_l^2(y)/y^2, & q = p_\perp \end{cases} \tag{8.12}$$

In the sum (8.10) ω takes discrete set of values determined by the equations (8.7) and (8.8). The relation (8.11) corresponds to the zero trace of the EMT.

The vev (8.10) are infinite. The renormalization of $\langle 0|T_{ik}|0\rangle$ in flat spacetime is affected by subtracting from this quantity its singular part $\langle \bar{0}|T_{ik}|\bar{0}\rangle$, which is precisely the value it would have if the boundary were absent. Here $|\bar{0}\rangle$ is the amplitude for Minkowski vacuum state. To evaluate the finite difference between these two infinities we will introduce a cutoff function $\psi_\mu(\omega)$, which decreases with increasing ω and satisfies the condition $\psi_\mu(\omega) \to 1$, $\mu \to 0$, and makes the sums finite. After subtracting we will allow $\mu \to 0$ and will show that the result does not depend on the form of cutoff:

$$\mathrm{reg}\langle 0|T_{ik}|0\rangle = \lim_{\mu\to 0}\left[\langle 0|T_{ik}|0\rangle - \langle \bar{0}|T_{ik}|\bar{0}\rangle\right]. \tag{8.13}$$

Hence we consider the following finite quantities

$$q(\mu,a,r) = \frac{1}{4\pi^2 a^4}\sum_{l=1}^{\infty}\sum_{k=1}^{\infty}\sum_{\lambda=0}^{1} j_{\nu,k}^{(\lambda)3} T_\nu(j_{\nu,k}^{(\lambda)})\psi_\mu(j_{\nu,k}^{(\lambda)}/a) D_l^{(q)}(j_{\nu,k}^{(\lambda)}x), \quad x=r/a, \tag{8.14}$$

where $\omega = j_{\nu,k}^{(\lambda)}/a$ are solutions to the eigenvalue equations (8.7) and (8.8) for $\lambda = 0, 1$, respectively. The summations over k in (8.14) can be done by using the formula (3.22) and taking $A=1, B=0$ for TM-modes ($\lambda=0$) and $A=1, B=2$ for TE-modes ($\lambda=1$) (recall that in (3.18) $\lambda_{\nu,k}$ are zeros of $\bar{J}_\nu(z)$ with bared quantities defined as (3.1)). Note that the resulting sums are of type (3.26). Let us substitute in formula (3.22)

$$f(z) = z^3 \psi_\mu(z/a) D_l^{(q)}(zx), \tag{8.15}$$

with $D_l^{(q)}(y)$ defined from (8.12). We will assume the class of cutoff functions for which the function (8.15) satisfies conditions for Theorem 2, uniformly with respect to μ (the corresponding restrictions for ψ_μ can be easily found from these conditions using the asymptotic formulae for Bessel functions). Below for simplicity we will consider the functions with no poles. In this case (8.15) is analytic on the right-half plane of the complex variable z. The discussion on the conditions to cutoff functions under which the difference between divergent sum and integral exists and has a finite value independent any further details of cutoff function see [47]. For TE- and TM-modes by choosing the constants A and B as mentioned above one obtaines

$$q = \frac{1}{8\pi^2}\sum_{l=1}^{\infty}\left\{2\int_0^\infty \omega^3\psi_\mu(\omega) D_l^{(q)}(\omega r)d\omega - \frac{1}{a^4}\int_0^\infty \chi_\mu(z/a) F_l^{(q)}(z,x)dz\right\}, \quad q=\varepsilon, p_\perp, p, \tag{8.16}$$

where for $x<1$ the functions $F_l^{(q)}(z,x)$ are defined as

$$
\begin{aligned}
F_l^{(\varepsilon)}(z,x) &= \frac{z}{x^2}\left[\frac{e_l(z)}{s_l(z)}+\frac{e_l'(z)}{s_l'(z)}\right]\left[ls_{l+1}^2(zx)+(l+1)s_{l-1}^2(zx)-(2l+1)s_l^2(zx)\right], &(8.17)\\
F_l^{(p_\perp)}(z,x) &= (2l+1)\frac{l(l+1)}{zx^4}\left[\frac{e_l(z)}{s_l(z)}+\frac{e_l'(z)}{s_l'(z)}\right]s_l^2(zx), &(8.18)\\
F_l^{(p)}(z,x) &= F_l^{(\varepsilon)}-2F_l^{(p_\perp)}, \quad \chi_\mu(y)=\left[\psi_\mu(iy)+\psi_\mu(-iy)\right]/2. &(8.19)
\end{aligned}
$$

In these expressions we have introduced Ricatti-Bessel functions of imaginary argument,

$$
s_l(z)=\sqrt{\frac{\pi z}{2}}I_\nu(z), \quad e_l(z)=\sqrt{\frac{2z}{\pi}}K_\nu(z), \, \nu=l+1/2. \tag{8.20}
$$

As $\langle\bar{0}|T_{ik}|\bar{0}\rangle=\lim_{a\to\infty}\langle 0|T_{ik}(\mu,a,r)|0\rangle$ the first integral in (8.16) represents the vacuum EMT for empty Minkowski spacetime:

$$
q=\frac{1}{4\pi^2}\sum_{l=1}^{\infty}\int_0^\infty \omega^3\psi_\mu(\omega)D_l^{(q)}(\omega r)d\omega. \tag{8.21}
$$

This expression can be further simplified. For example in the case of the energy density one has

$$
\begin{aligned}
\varepsilon^{(0)} &= \frac{1}{4\pi^2}\sum_{l=1}^{\infty}\int_0^\infty \omega^3\psi_\mu(\omega)\left[lj_{l+1}^2(\omega r)+(l+1)j_{l-1}^2(\omega r)+(2l+1)j_l^2(\omega r)\right]d\omega=\\
&= \frac{1}{2\pi^2}\int_0^\infty \omega^3\psi_\mu(\omega)\sum_{l=0}^{\infty}(2l+1)j_l^2(\omega r)d\omega=\int_0^\infty \omega^3\psi_\mu(\omega)d\omega. \qquad (8.22)
\end{aligned}
$$

As we see the using of GAPF allows us to extract from infinite quantities the divergent part without specifying the form of cutoff function. Now the regularization of the EMT is equivalent to the omitting the first summand in (8.16), which as we saw corresponds to the contribution of the spacetime without boundaries. For the regularized components one obtains

$$
\mathrm{reg}\, q(a,r)=-\frac{1}{8\pi^2a^4}\sum_{l=1}^{\infty}\int_0^\infty \chi_\mu(z/a)F_l^{(q)}(z,x)dz, \quad r<a, \quad q=\varepsilon,p,p_\perp. \tag{8.23}
$$

By using the recurrence relations for Riccati-Bessel modified functions the expressions for the regularized vacuum energy density and radial pressure may be presented in the form

$$
q=\frac{-1}{8\pi^2a^2r^2}\sum_{l=1}^{\infty}(2l+1)\int_0^\infty dzz\chi_\mu(z/a)\left[\frac{e_l(z)}{s_l(z)}+\frac{e_l'(z)}{s_l'(z)}\right]\left\{s_l'^2(zx)-s_l^2(zx)\left[1-\frac{l(l+1)}{(-1)^iz^2x^2}\right]\right\}, \tag{8.24}
$$

where $i=0$ for $q=\varepsilon$ and $i=1$ for $q=p$. Here we keep the cutoff factor because it plays an important role in the calculations of the total Casimir energy for the spherical shell (see below). The derivation of the vacuum densities (8.23), (8.24) given above uses GAPF to summarize mode sums and is based on [13, 14]. One can see that this formulae for the case of exponential cutoff function may be obtained also from the results of [36, 37], where Green function method is used.

We obtained the regularized values (8.23) by introducing a cutoff function and susequent subtracting the contribution due to the unbounded space. The GAPF in the form (3.22) allows to obtain immediately this finite difference. However it should be noted that by using GAPF in the form (3.18) we can derive the expressions for the regularized azimuthal pressure without introducing any special cutoff function. To see this note that for $x<1$ the function (8.15) with

$q = p_\perp$ and $\psi_\mu = 1$ satisfies conditions to Theorem 2. It follows from here that we can apply the formula (3.18) directly to the corresponding sum over ω in (8.10) or over k in (8.14) (with $\psi_\mu = 1$) without introducing the cutoff function. This immediately yields to the formula (8.23) for $q = p_\perp$ with $\psi_\mu = 1$.

Let us consider the behaviour of the functions $F_l^{(q)}(z,x)$ in various limiting cases. By using the corresponding formulae for Bessel functions one obtains:

(a) When l is fixed and $z \to 0$

$$F_l^{(\varepsilon)}(z,x) \sim \pi(l+1/2)x^{2(l-1)}, \quad F_l^{(p_\perp)}(z,x) \sim \pi l x^{2(l-1)}/2. \tag{8.25}$$

(b) When l is fixed and zx is large

$$F_l^{(\varepsilon)} \sim 2F_l^{(p_\perp)} \sim l^2(l+1)^2(2l+1)\frac{\pi}{2(zx)^4}\exp[-2z(1-x)], \tag{8.26}$$

and the integral over z in (8.23) converges for all $x \le 1$;

(c) For large l by using the uniform asymptotic expansions for Bessel functions [21] one finds

$$F_l^{(q)}(\nu z,x) \sim \Phi_l^{(q)}(z,x)\exp\left\{-2\nu[\eta(z)-\eta(zx)]\right\}, \quad \nu = l+1/2 \tag{8.27}$$

with

$$\Phi_l^{(\varepsilon)}(z,x) = \frac{\pi\nu}{x^3}t(zx)t^3(z)\left\{1-\frac{1}{12\nu}\left[t(z)(t^2(z)+3)+t(zx)(5t^2(zx)-9)\right]\right\}, \tag{8.28}$$

$$\Phi_l^{(p)}(z,x) = \frac{\pi}{2x^3}t^2(zx)t^3(z), \tag{8.29}$$

where the standard notations are used:

$$t(z) = \frac{1}{\sqrt{1+z^2}}, \quad \eta(z) = \sqrt{1+z^2} + \ln\frac{z}{1+\sqrt{1+z^2}}. \tag{8.30}$$

From these asymptotic formulae it follows that in (8.23) and (8.24) the rhs is finite for $x < 1$ and the cutoff may be removed by putting $\chi_\mu = 1, \mu \to 0$. From here it is obvious the independence of the regularized quantities on the specific form of the cutoff, on class of functions for which (8.15) satisfy conditions for (3.22).

From (8.27), (8.28) and (8.29) it follows that the vev of the EMT diverge at sphere surface, $x \to 1$, due to the contribution of large l (note that, as it follows from (8.28) and (8.29), in (8.23) with $\chi_\mu = 1$ the integral over z converges at $x = 1$). The corresponding asymptotic behaviour can be found by using the uniform asymptotic expansions given above and the leading terms have the form

$$\varepsilon \sim 2p_\perp \sim \frac{-1}{30\pi^2 a(a-r)^3}, \quad p \sim \frac{-1}{60\pi^2 a^2(a-r)^2}. \tag{8.31}$$

These surface divergences originate in the unphysical nature of perfect conductor boundary conditions and are well known in quantum field theory with boundaries. They are investigated in detail for various types of fields and general shape of the boundary [45, 46]. Eqs. (8.31) are particular cases of the asymptotic expansions for EMT vev near the smooth boundary given in these papers. In reality the expectation values for the EMT components will attain a limiting value on the conductor surface, which will depend on the molecular details of the conductor. From the asymptotic expansions given above it follows that the main contributions to $q(r)$ are due to the frequencies $\omega < (a-r)^{-1}$. Hence we expect that the formulae (8.23) are valid for real conductors up to distances r for which $(a-r)^{-1} \ll \omega_0$, with ω_0 being the characteristic frequency, such that for $\omega > \omega_0$ the conditions for perfect conductivity are failed.

At the sphere centre in (8.23) $l=1$ multipole contributes only and we obtain [34, 14]

$$\begin{aligned}\varepsilon(0) &= -\frac{1}{2\pi^2 a^4}\int_0^\infty dz z^3\left[\left(\frac{z-1}{z+1}e^{2z}+1\right)^{-1}-\left(\frac{z^2-z+1}{z^2+z+1}e^{2z}-1\right)^{-1}\right]=-0.0381a^{-4},\\ p(0) &= p_\perp(0)=\varepsilon(0)/3.\end{aligned} \tag{8.32}$$

At centre the equation of state for the electromagnetic vacuum is the same as that for blackbody radiation. Note that the corresponding results obtained using the uniform asymptotic expansions for Bessel functions [36, 37] are in good agreement with (8.32).

The components of the regularized EMT satisfy continuity equation $T^k_{i;k}=0$, which for the spherical geometry takes the form

$$p'(r)+\frac{2}{r}(p-p_\perp)=0. \tag{8.33}$$

From here by using the zero trace condition the following integral relations may be obtained

$$p(r)=\frac{1}{r^3}\int_0^r\varepsilon(t)t^2dt=\frac{2}{r^2}\int_0^r p_\perp(t)tdt, \tag{8.34}$$

where the integration constant is determined from the relations (8.32) at the sphere centre. It follows from the first relation that the total energy within a sphere with radius r is equal to

$$E(r)=4\pi\int_0^r\varepsilon(t)t^2dt=3V(r)p(r), \tag{8.35}$$

where $V(r)$ is the corresponding volume.

The distribution for the vacuum energy density and pressures inside the perfectly conducting sphere can be obtained from the results of the numerical calculations given in [36, 37]. In their calculations Brevik and Kolbenstvedt use the uniform asymptotic expansions of Ricatti-Bessel functions for large values of order. In [13, 14] (see also [12]) the corresponding quantities are calculated on the base of the exact relations for these functions and the accuracy of the numerical results in [36, 37] is estimated ($\approx 5\%$). The simple approximataion formulae are presented with the same accuracy as asymptotic expressions. Note that inside the sphere all quantities ε, p, $p_\perp$ are negative and corresponding vacuum forces tend to contract sphere.

9 Electromagnetic vacuum EMT outside a spherical shell

Now let us consider the electromagnetic vacuum in the region outside of a perfectly conducting sphere. To deal with discrete modes we firstly consider vacuum fields in the region between two cocentric conducting spherical shells with radii a and b, $a<b$. Letting $b\to\infty$ we will obtain from here the result for the region under question.

By using Coulomb gauge the complete set of solutions to the Maxwell equations can be written in the form similar to (8.2)

$$\mathbf{A}_{\omega lm\lambda}(\mathbf{r},t)=\frac{e^{-i\omega t}}{\sqrt{4\pi}}\beta_{\lambda l}(a,b,\omega)\left\{\begin{array}{ll}\omega g_{0l}(\omega a,\omega r)\mathbf{X}_{lm} & \text{if}\quad\lambda=0\\ \nabla\times[g_{1l}(\omega a,\omega r)\mathbf{X}_{lm}] & \text{if}\quad\lambda=1\end{array}\right., \tag{9.1}$$

where as above the values $\lambda=0$ and $\lambda=1$ correspond to the waves of magnetic (TE-modes) and electric (TM-modes) type,

$$g_{\lambda l}(x,y)=\left\{\begin{array}{ll}j_l(y)n_l(x)-j_l(x)n_l(y) & \text{if}\quad\lambda=0\\ j_l(y)[xn_l(x)]'-[xj_l(x)]'n_l(y), & \text{if}\quad\lambda=1\end{array}\right., \tag{9.2}$$

with $n_l(x)$ being Neumann spherical function. From the standard boundary conditions at surfaces $r = a$ and $r = b$ one finds that possible energy levels of photon are solutions to the following equations

$$\left(\frac{d}{dr}\right)^{\lambda} \left[r g_{\lambda l}(\omega a, \omega r)\right]_{r=b} = 0, \quad \lambda = 0, 1. \tag{9.3}$$

All roots of these equations are real and simple [21].

The coefficients $\beta_{\lambda l}$ in (9.1) are determined from the normalization condition (8.4), where now the integration goes over the region between spherical shells, $a \le r \le b$. By using the standard relations for spherical Bessel functions they can be presented in the form

$$\beta_{0l}^2 = \omega a \left[\frac{a j_l^2(\omega a)}{b j_l^2(\omega b)} - 1\right]^{-1}, \quad \lambda = 0 \tag{9.4}$$

$$\beta_{1l}^2 = \frac{1}{\omega a} \left\{\frac{b\left[\omega a j_l(\omega a)\right]'^2}{a\left[\omega b j_l(\omega b)\right]'^2}\left[1 - \frac{l(l+1)}{\omega^2 b^2}\right] - 1 + \frac{l(l+1)}{\omega^2 a^2}\right\}^{-1}, \quad \lambda = 1. \tag{9.5}$$

From (8.1) with the electromagnetic field EMT and functions (9.1) as a complete set of solutions one obtains the vev in the form (8.9) with

$$q(a,b,r) = \frac{1}{8\pi} \sum_{\omega l \lambda} (2l+1)\omega^4 \beta_{\lambda l}^2 f_{\lambda l}^{(q)}(\omega a, \omega r), \quad q = \varepsilon,\, p,\, p_\perp, \tag{9.6}$$

where the frequencies ω are solutions to the equations (9.3), and

$$f_{\lambda l}^{(\varepsilon)}(\omega a, \omega r) = \left[1 + \frac{l(l+1)}{\omega^2 r^2}\right] g_{\lambda l}^2(\omega a, \omega r) + \frac{1}{\omega^2 r^2}\left[\frac{d}{d(\omega r)}\left(\omega r g_{\lambda l}(\omega a, \omega r)\right)\right]^2, \tag{9.7}$$

$$f_{\lambda l}^{(p_\perp)}(\omega a, \omega r) = \frac{l(l+1)}{\omega^2 r^2} g_{\lambda l}^2(\omega a, \omega r). \tag{9.8}$$

It is easy to see that the eigenvalue equations (9.3) can be written in terms of the function C_ν^{AB}, defined by (4.1), as

$$C_\nu^{AB}(\eta, \omega a) = 0, \quad \nu = l + 1/2,\ \eta = b/a,\ A = 1/(1+\lambda),\ B = \lambda,\ \lambda = 0, 1. \tag{9.9}$$

By this choice of constants A and B the normalization coefficients (9.4) and (9.5) are related with the function T_ν^{AB} from (4.6) as:

$$\beta_{\lambda l}^2 = T_\nu^{AB}(\eta, \omega a). \tag{9.10}$$

This allows to use the formulae from section 4 for the summation over eigenmodes.

As above to regularize the infinite quantities (9.6) we introduce a cutoff function $\psi_\mu(\omega)$ and consider the difference

$$\mathrm{reg}\langle 0|T_{ik}|0\rangle = \lim_{\mu\to 0}\left[\langle 0|T_{ik}(\mu,a,b)|0\rangle - \lim_{a\to 0}\lim_{b\to\infty}\langle 0|T_{ik}(\mu,a,b)|0\rangle\right]. \tag{9.11}$$

This procedure is equivalent to the subtraction of Minkowskian part without boundaries.

Hence instead of (9.6) we consider the finite quantities

$$q = \frac{1}{8\pi a^4}\sum_{l=1}^{\infty}(2l+1)\sum_{k=1}^{\infty}\sum_{\lambda=0}^{1}\gamma_{\nu,k}^{(\lambda)4} T_\nu^{AB}(\eta, \gamma_{\nu,k}^{(\lambda)})\psi_\mu(\gamma_{\nu,k}^{(\lambda)}/a) f_{\lambda l}^{(q)}(\gamma_{\nu,k}^{(\lambda)}, \gamma_{\nu,k}^{(\lambda)}x), \quad q = \varepsilon, p_\perp, \tag{9.12}$$

where $x = r/a$, and $\omega a = \gamma_{\nu,k}^{(\lambda)}$ are solutions to the equations (9.3) or (9.9). To sum over k we will use the formula (4.13) with

$$h(z) = z^4 \psi_\mu(z/a) f_{\lambda l}^{(q)}(z, zx), \tag{9.13}$$

assuming a class of cutoff functions for which (9.13) satisfies to the conditions (4.4) and (4.11) uniformly with respect to μ. The corresponding restrictions on ψ_μ can be obtained using the asymptotic formulae for Bessel functions. From (9.12) by applying to the sum over k the formula (4.13) for the EMT components one obtains

$$\begin{aligned} q &= \frac{1}{8\pi^2 a^4} \sum_{l=1}^{\infty} (2l+1) \sum_{\lambda=0}^{1} \left\{ \int_0^\infty z^3 \psi_\mu(z/a) \frac{f_{\lambda l}^{(q)}(z, zx)}{\Omega_{1\lambda l}(z)} dz + \right. \\ &\quad \left. + \frac{1}{x^2} \int_0^\infty \frac{e_l^{(\lambda)}(\eta z)}{e_l^{(\lambda)}(z)} \frac{z \chi_\mu(z/a) F_{\lambda l}^{(q)}(z, zx)}{[(\partial/\partial y)^\lambda G_{\lambda l}(z, y)]_{y=z\eta}} dz \right\}, \end{aligned} \tag{9.14}$$

where we use the notations

$$e_l^{(\lambda)}(y) \equiv \left(\frac{d}{dy}\right)^\lambda e_l(y), \quad s_l^{(\lambda)}(y) \equiv \left(\frac{d}{dy}\right)^\lambda s_l(y) \tag{9.15}$$

for the Riccati-Bessel functions derivatives,

$$\Omega_{1\lambda l}(z) = \begin{cases} j_l^2(z) + n_l^2(z), & \lambda = 0 \\ [z j_l(z)]'^2 + [z n_l(z)]'^2, & \lambda = 1 \end{cases}, \tag{9.16}$$

and

$$\begin{aligned} G_{\lambda l}(x, y) &= e_l^{(\lambda)}(x) s_l(y) - e_l(y) s_l^{(\lambda)}(x), \quad \lambda = 0, 1, && (9.17) \\ F_{\lambda l}^{(\varepsilon)}(z, y) &= \left[\frac{\partial}{\partial y} G_{\lambda l}(z, y)\right]^2 + \left[\frac{l(l+1)}{y^2} - 1\right] G_{\lambda l}^2(z, y), && (9.18) \\ F_{\lambda l}^{(p_\perp)}(z, y) &= l(l+1) G_{\lambda l}^2(z, y)/y^2. && (9.19) \end{aligned}$$

The function χ_μ is determined by (8.18). To obtain the vev for the EMT components outside of a single conducting spherical shell with radius a let us consider the limit $b \to \infty$. In this limit the second integral on the right of formula (9.14) tends to zero (for large $\eta = b/a$ the subintegrand is proportional to $e^{-2\eta z}$), whereas the first one does not depend on b. Hence one obtains

$$q = \frac{1}{8\pi^2 a^4} \sum_{l=1}^{\infty} (2l+1) \sum_{\lambda=0,1} \int_0^\infty z^3 \psi_\mu(z/a) \frac{f_{\lambda l}^{(q)}(z, zx)}{\Omega_{1\lambda l}(z)} dz, \quad q = \varepsilon, p_\perp. \tag{9.20}$$

To regularize the expressions (9.20) we have to subtract the Minkowskian part, namely the expression (8.21). It can be easily seen that

$$\frac{f_{\lambda l}^{(q)}(z, zx)}{\Omega_{1\lambda l}(z)} - D_l^{(q)}(zx) = -\frac{1}{2} \sum_{m=1,2} \Omega_{\lambda l}^{(m)}(z) D_l^{(mq)}(zx). \tag{9.21}$$

Here the functions $D_l^{(mq)}(y)$ are obtained from the relations (8.12) by replacing $j_l \to h_l^{(m)}$, with $h_l^{(m)}$, $m = 1, 2$ being spherical Hankel functions, and

$$\Omega_{\lambda l}^{(m)}(z) = \begin{cases} j_l(z)/h_l^{(m)}(z), & \lambda = 0 \\ [z j_l(z)]'/[z h_l^{(m)}(z)]', & \lambda = 1 \end{cases} \tag{9.22}$$

The function $h_l^{(1)}(z)k$ $(h_l^{(2)}(z))$ has no zeros for $0 \leq \arg z \leq \pi/2$ $(-\pi/2 \leq \arg z \leq 0)$ and from this it follows that

$$\sum_{m=1,2} \int_0^\infty z^3 \psi_\mu(z/a) \Omega_{\lambda l}^{(m)}(z) D_l^{(mq)}(zx) dz = 2 \int_0^\infty z^3 \mathrm{Re} \left[\psi_\mu(iz/a) \Omega_{\lambda l}^{(1)}(iz) D_l^{(1q)}(izx) \right] dz. \quad (9.23)$$

By introducing Ricatti-Bessel functions (8.20), for the regularized components of the vacuum EMT outside the sphere we find

$$q(a,r) = -\frac{1}{8\pi^2 a^4} \sum_{l=1}^{\infty} \int_0^\infty \chi_\mu(z/a) F_l^{(q)}(z,x) dz, \quad r > a, \quad q = \varepsilon, p_\perp, p, \quad (9.24)$$

where for $x > 1$ the functions $F_l^{(q)}(z,x)$ are defined as

$$F_l^{(\varepsilon)}(z,x) = \frac{z}{x^2} \left[\frac{s_l(z)}{e_l(z)} + \frac{s_l'(z)}{e_l'(z)} \right] \left[l e_{l+1}^2(zx) + (l+1) e_{l-1}^2(zx) - (2l+1) e_l^2(zx) \right] \quad (9.25)$$

$$F_l^{(p_\perp)}(z,x) = (2l+1) \frac{l(l+1)}{zx^4} \left[\frac{s_l(z)}{e_l(z)} + \frac{s_l'(z)}{e_l'(z)} \right] e_l^2(zx), \quad F_l^{(p)} = F_l^{(\varepsilon)} - 2F_l^{(p_\perp)}. \quad (9.26)$$

The exterior mode sum consideration given in this section follows [13, 15]. For the case of exponential cutoff function the formulae (9.24) and (9.26) can be obtained also from the results [36, 37], where Green's function formalism was used. Note that the expressions for the exterior components are obtained from the interior ones replacing $s_l \to i_l$, $i_l \to s_l$. In particular, the exterior components can be presented in the form analog to (8.24).

Let us consider the behaviour of the functions $F_l^{(q)}(z,x)$ in various limiting cases. By using the corresponding formulae for Bessel functions one obtains:

(a) When l is fixed and $z \to 0$

$$F_l^{(\varepsilon)}(z,x) \sim -\pi(l+1/2) x^{-2(l+2)}, \quad F_l^{(p_\perp)}(z,x) \sim -\pi(l+1) x^{-2(l+1)}/2. \quad (9.27)$$

(b) When l is fixed and z is large

$$F_l^{(\varepsilon)} \sim 2F_l^{(p_\perp)} \sim -l^2(l+1)^2(2l+1) \frac{\pi}{2(zx)^4} \exp[-2z(x-1)]; \quad (9.28)$$

(c) For large l by using the uniform asymptotic expansions for Bessel functions [21] one finds

$$F_l^{(q)}(\nu z, x) \sim \Phi_l^{(q)}(z,x) \exp\left\{-2\nu[\eta(zx) - \eta(z)]\right\}, \quad \nu = l + 1/2 \quad (9.29)$$

with

$$\Phi_l^{(\varepsilon)}(z,x) = -\frac{\pi\nu}{x^3} t(zx) t^3(z) \left\{ 1 + \frac{1}{12\nu} \left[t(z)(t^2(z)+3) + t(zx)(5t^2(zx) - 9) \right] \right\}, \quad (9.30)$$

$$\Phi_l^{(p)}(z,x) = \frac{\pi}{2x^3} t^2(zx) t^3(z), \quad (9.31)$$

whith notations (8.30).

It follows from here that for the values $x > 1$ the expressions (9.24) are finite and hence cutoff may be removed. In this case the independence of the result on specific form of cutting function is obvious.

The expressions (9.24) with $\psi_\mu = 1$ diverge at sphere surface. The leading terms of these divergences may be found using (9.29) and are as following

$$\varepsilon \sim 2p_\perp \sim \frac{1}{30\pi^2 a(r-a)^3}, \quad p(r) \sim \frac{-1}{60\pi^2 a^2 (r-a)^2}, \quad (9.32)$$

and

$$\lim_{r\to a}(\varepsilon/p_\perp)' = \lim_{r\to a}(p/p_\perp)' = -1. \qquad (9.33)$$

Comparing (9.32) with (8.31) we see that the cancellation of interior and exterior leading divergent terms occurs in calculating the total energy and force acting on sphere. The same cancellations take place for the next subleading divergent terms as well (see below). Formulas (9.32) are particular cases of general asymptotic expansions of the vacuum EMT components for conformally invariant fields near an arbitrary smooth boundary given in [45].

For distances far from the sphere one finds

$$p_\perp \sim \frac{1}{4\pi^2 a^4 x^7}\int_0^\infty z^2 e_1^2(z)dz = \frac{5a^3}{16\pi^2 r^7}, \quad \varepsilon \sim -4p \sim \frac{a^3}{2\pi^2 r^7}, \quad r \gg a. \qquad (9.34)$$

The results of numerical calculations of the vacuum EMT components outside the sphere are given in [36, 37, 14]. In [36, 37] calculations are carried out by using the uniform asymptotic expansions for Riccati-Bessel functions. The accuracy of this approximation is estimated in [14], where exact relations are used in numerical calculations. The simple approximating formulas with the same accuracy as those for the asymptotic calculations are presented as well. The energy density and azimuthal pressure are positive, and radial pressure is negative. The latter means that the exterior vacuum forces tend to expand sphere. As we will see below this dominates the interior contraction force.

Note that the continuity equation (8.33) now may be written in the following integral form

$$p(r) = \frac{1}{r^3}\int_\infty^r \varepsilon(t)t^2 dt = \frac{2}{r^2}\int_\infty^r p_\perp(t)t dt, \qquad (9.35)$$

where the integration constant is determined from the asymptotic relations (9.32). From (8.34) and (9.35) it follows that

$$E(a) = \int \varepsilon(r)dV = 4\pi a^3[p(a-) - p(a+)], \qquad (9.36)$$

where $E(a)$ is the total vacuum energy for a spherical shell with radius a, $p(a\pm) = \lim_{r\to 0} p(a\pm r)$. By using the expressions for $p(r)$ given above one can obtain the following formula for the total energy (the same result can be obtained also by integrating the energy density)

$$\begin{aligned} E(a) &= \frac{-1}{2\pi a}\sum_{l=1}^\infty (2l+1)\int_0^\infty dz\chi_\mu(z/a)z\left(\ln|s_l(z)e_l(z)|\right)'\left[1+\left(\frac{l(l+1)}{z^2}+1\right)\frac{s_l(z)e_l(z)}{s_l'(z)e_l'(z)}\right] = \\ &= \frac{-1}{2\pi a}\sum_{l=1}^\infty (2l+1)\int_0^\infty dz\chi_\mu(z/a)z\frac{d}{dz}\ln\left\{1-[s_l(z)e_l(z)]'^2\right\}. \end{aligned} \qquad (9.37)$$

By taking the cutting function $\psi_\mu(\omega) = e^{-\mu\omega}$ one obtains the expression for the Casimir energy of the sphere derived in [33] by Green function method. Note that in this method the factor $\psi_\mu(iz/a) = e^{-i\omega\mu}$ appears automatically as a result of the point splitting procedure. The evaluation of (9.37) leads to the result $E = 0.092353/2a$ for the Casimir energy of a spherical conducting shell [30, 31, 32, 33, 40, 38]. This corresponds to the repulsive vacuum force on the sphere. Here the cancellation of interior and exterior divergent terms in the energy density occurs. The discussion on cancellations of divergences between interior and exterior modes see [33, 45, 48, 36].

10 Electromagnetic vacuum in spherical layer between perfectly conducting surfaces

Electromagnetic vev of the EMT in the region between two cocentric perfectly conducting surfaces with radii a and b, $a < b$, may be obtained from the results of previous section. The corresponding nonrenormalized components are given by (9.14). Using this formula they can be presented in the form

$$q(a,b,r) = q(a,r) + q^{(ab)}(r), \quad a < r < b, \quad q = \varepsilon, p_\perp, p, \quad p = \varepsilon - 2p_\perp, \tag{10.1}$$

where $q(a,r)$ is given by (9.20), and [16]

$$q^{(ab)}(r) = \frac{1}{8\pi^2 a^2 r^2} \sum_{l=1}^{\infty} (2l+1) \sum_{\lambda=0,1} \int_0^\infty z\psi_\mu(z/a)\Omega_{\lambda l}(z,\eta) F_{\lambda l}^{(q)}(z, zx)dz, \quad x = r/a. \tag{10.2}$$

Here the functions $F_{\lambda l}^{(q)}$ are defined by relations (9.18), (9.19), $\eta = b/a$, and

$$\Omega_{\lambda l}(z,\eta) = \frac{e_l^{(\lambda)}(z\eta)/e_l^{(\lambda)}(z)}{e_l^{(\lambda)}(z)s_l^{(\lambda)}(z\eta) - e_l^{(\lambda)}(z\eta)s_l^{(\lambda)}(z)} \tag{10.3}$$

(see notation (9.15)). In (10.1) the dependence on b is contained in the summand $q^{(ab)}$ only. This quantity is finite for $a \le r < b$ and the regularization of $q(a,b,r)$ is equivalent to the renormalization of the first summand. This procedure have been done in previous section, where we have seen that $q(a,r)$ (see expressions (9.24) and (9.25)) coincides with the corresponding quantity for the exterior region of a single shell with radius a. The expressions (10.2) for $a \le r < b$ are finite when $\mu \to 0$ and hence for these values the cuttoff function may be removed putting $\chi_\mu = 1$.

It can be seen that the quantities (10.1) may be written also in the form

$$q(a,b,r) = q(b,r) + \tilde{q}^{(ab)}(r), \quad q = \varepsilon, p_\perp, p, \tag{10.4}$$

where

$$\tilde{q}^{(ab)}(r) = \frac{1}{8\pi^2 b^2 r^2} \sum_{l=1}^{\infty} (2l+1) \sum_{\lambda=0,1} \int_0^\infty z\tilde{\Omega}_{\lambda l}(z,\sigma) F_{\lambda l}^{(q)}(z, zy)dz, \tag{10.5}$$

with $y = r/b$, $\sigma = a/b$, and

$$\tilde{\Omega}_{\lambda l}(z,\sigma) \equiv \frac{s_l^{(\lambda)}(z\sigma)/s_l^{(\lambda)}(z)}{e_l^{(\lambda)}(z\sigma)s_l^{(\lambda)}(z) - e_l^{(\lambda)}(z)s_l^{(\lambda)}(z\sigma)}. \tag{10.6}$$

In (10.4) $\tilde{q}^{(ab)}(r) \to 0$ when $a \to 0$ and $q(b,r)$ coincides with the corresponding quantities inside a single conducting shell with radius b (the latter can be seen also by direct evaluation of $q(b,r)$). Note that in (10.5) the sum and integral are convergent for $a < r \le b$.

As we said above from the expressions $q(a,b,r)$ in limiting cases $a \to 0$ or $b \to \infty$ may be obtained the vacuum stress inside and outside of a single shell. Consider now the another limiting case: $h = b - a = const$, $b \to \infty$. For $a/b \to 1$ the main contribution in (10.2) is due to large l. This allows us to use asymptotic formulae for Bessel functions. For instance, in the case of the energy density one has

$$\varepsilon \approx \varepsilon^{(ab)} \approx -\frac{1}{\pi^2 b^4} \int_0^\infty z^2 \frac{\Lambda(z, a/b)}{\sqrt{1+z^2}} dz, \tag{10.7}$$

where

$$\Lambda = \sum \nu^3 \left\{ e^{2\nu[\eta(z)-\eta(za/b)]} - 1 \right\}^{-1} \approx \frac{b^4}{16h^4(1+z^2)^2} \int_0^\infty \frac{s^3 ds}{e^s - 1} = \frac{\pi^4 b^4}{240h^4(1+z^2)^2}. \tag{10.8}$$

By substituting this into (10.7) we receive the standard result for the Casimir parallel plate configuration: $\varepsilon = -\pi^2/720h^4$.

Let us present the quantites $q = \varepsilon, p, p_\perp$ in the form

$$q = q(a,r) + q(b,r) + \Delta q(a,b,r), \quad a < r < b, \tag{10.9}$$

where "interference" term may be written in two ways

$$\Delta q(a,b,r) = q^{(ab)}(a,b,r) - q(b,r) \tag{10.10}$$
$$\Delta q(a,b,r) = \tilde{q}^{(ab)}(a,b,r) - q(a,r). \tag{10.11}$$

Here $q^{(ab)}$ and $\tilde{q}^{(ab)}$ are defined by relations (10.2) and (10.5). It can be seen that $\Delta q(a,b,r)$ is finite for all $a \le r \le b$, $a < b$. Near the surface $r = a$ it is convenient to use (10.10), as for $r \to a$ both summands in this formula are finite. For the same reason the formula (10.11) is convenient for calculations near the surface $r = b$.

So far we have considered the electromagnetic vacuum in the region between two perfectly conducting spherical surfaces. Consider now a system consisting two cocentric thin spherical shells with radii a and b, $a < b$. In this case the vev for the EMT components may be written in the form

$$q(a,b,r) = q(a,r)\theta(a-r) + q(b,r)\theta(r-b) + \left[q(a,r) + q^{(ab)}(r) \right] \theta(r-a)\theta(b-r), \tag{10.12}$$

where $\theta(x)$ is the unit step function. By using the continuity equation (8.33) it is easy to see that the total Casimir energy for the system under consideration can be presentaed in the form

$$E^{(ab)} = E(a) + E(b) + 4\pi \left[b^3 \tilde{p}^{(ab)}(b) - a^3 p^{(ab)}(a) \right], \tag{10.13}$$

where $E(i)$ is the Casimir energy for a single sphere with radius i, $i = a, b$. As it follows from (10.2) and (10.5) the additional vacuum pressures on the spheres are equal to [16, 12]

$$p^{(ab)}(a) = \frac{1}{8\pi^2 a^4} \sum_{l=1}^{\infty} (2l+1) \int_0^\infty dz z \left\{ \left[\frac{l(l+1)}{z^2} + 1 \right] \Omega_{1l}(z,\eta) - \Omega_{0l}(z,\eta) \right\} \tag{10.14}$$
$$\tilde{p}^{(ab)}(b) = \frac{1}{8\pi^2 b^4} \sum_{l=1}^{\infty} (2l+1) \int_0^\infty dz z \left\{ \left[\frac{l(l+1)}{z^2} + 1 \right] \tilde{\Omega}_{1l}(z,\sigma) - \tilde{\Omega}_{0l}(z,\sigma) \right\}, \tag{10.15}$$

where $\Omega_{\lambda l}$ and $\tilde{\Omega}_{\lambda l}$ are defined by relations (10.3) and (10.6).

The vacuum force per unit area of the inner sphere is equal to

$$F^{(a)} = F_1^{(a)} + \Delta F^{(a)}, \quad \Delta F^{(a)} = -p^{(ab)}(a) \tag{10.16}$$

where $F_1^{(a)}$ is the force per unit area of a single sphere with radius a, and $\Delta F^{(a)}$ is due to the existence of the second sphere ("interaction" force). By similar way vacuum force acting on per unit area of outer sphere is

$$F^{(b)} = F_1^{(b)} + \Delta F^{(b)}, \quad \Delta F^{(b)} = \tilde{p}^{(ab)}(b). \tag{10.17}$$

The results of numerical calculations of quantities $\Delta q(a,b,r)$, $q = \varepsilon, p, p_\perp$, as well as those for $\Delta F^{(a,b)}$ are presented in [16, 12]. Note that as it follows from the results of these calculations the quantities (10.14) and (10.15) are always negative, and therefore the interaction forces between two spheres are always attractive (as in the parallel plate configuration). The total Casimir energy is positive for small values of a/b and is negative for values close to 1. At $a/b \approx 0.7$ this energy is zero.

11 EMT vev inside a perfectly conducting cylindrical shell

In this and next sections we will consider the case of perfectly conducting cylindrically symmetric boundaries. The Casimir effect for a perfectly conducting cylindrical shell was considered in [49] (see also [50]) and for a dielectric cylinder in [51] by using the Green function formalism. Recently the problem is reconsidered in [52, 53] using the mode summation technique and in [42, 54, 55], within the framework of the zeta-function regularization scheme. In these papers global quantities, such as the total enehgy and stress on a shell, are investigated. Local characteristics of the electromagnetic vacuum are considered in [17] for the interior and exterior regions of a conducting cylindrical shell, and in [18] for two coaxial shells. In this papers the mode summation method is used combined with generalized Abel-Plana formula. Our consideration below is based on these works (see also [12]). The vev of the EMT for electromagnetic field inside a perfectly conducting cylindrical surface with radius a can be found by the way similar to the spherical case. As an eigenfunctions we use the vector potentials corresponding to the cylindrical waves of magnetic ($\lambda = 0$) and electric ($\lambda = 1$) type:

$$\mathbf{A}_\alpha = \beta_{\lambda m} \left\{ \begin{array}{ll} -\mathbf{e}_3 \times \nabla_t \left\{ J_m(\gamma r) \exp\left[i(m\varphi + kz - \omega t)\right]\right\}, & \lambda = 0 \\ (1/i\omega)\left[\mathbf{e}_3 + (ik/\gamma^2)\nabla_t\right] J_m(\gamma r) \exp\left[i(m\varphi + kz - \omega t)\right], & \lambda = 1 \end{array} \right. , \tag{11.1}$$

where the cylindrical coordinates (r, φ, z) are used with unit vectors $\mathbf{e}_i$, $\gamma^2 = \omega^2 - k^2$, m is an integer, ∇_t is the transverse to the z axis part of the nabla operator. From the standard boundary conditions we obtain the following equations for the possible values of the quantum number γ:

$$\begin{aligned} J_m'(\gamma a) &= 0, \quad \lambda = 0, \\ J_m(\gamma a) &= 0, \quad \lambda = 1. \end{aligned} \tag{11.2}$$

The constants $\beta_{\lambda m}$ are determined from the normalization condition and are equal to

$$\beta_{\lambda m}^2 = \frac{\gamma^2}{\pi \omega a^2} \left[J_m'^2(\gamma a) + (1 - m^2/\gamma^2 a^2) J_m^2(\gamma a) \right]^{-1} = \frac{\gamma^3}{\pi \omega a} T_m(\gamma a), \tag{11.3}$$

where $T_m(z)$ is defined by (3.8). As independent quantum numbers we will choose the set $\alpha = (mk\gamma\lambda)$. In this case $\omega^2 = \gamma^2 + k^2$ and γ takes discrete values being solutions to (11.2). By using the standard formula (8.1) with (11.1) one obtains

$$\langle 0 | T_k^i | 0 \rangle = \mathrm{diag}\left(\varepsilon, \, -p_1, \, -p_2, \, -p_3\right), \tag{11.4}$$

where the energy density ε, the pressure p_i in direction $\mathbf{e}_i$ can be presented in the form (below the index c will specify quantities for the cylindrical geometry)

$$q_c(a, r) = \frac{1}{8\pi} \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{+\infty} dk \sum_{\lambda, \gamma} \beta_{\lambda m}^2 f_m^{(q)}(\gamma r), \quad q_c = \varepsilon, p_1, p_2 \tag{11.5}$$

with

$$f_m^{(\varepsilon)}(y) = \left(\frac{2k^2}{\gamma^2} + 1\right) \left[J_m'^2(y) + \frac{m^2}{y^2} J_m^2(y) \right] + J_m^2(y) \tag{11.6}$$

$$f_m^{(p_i)}(y) = -(-1)^i \left[J_m'^2(y) - \left(\frac{m^2}{y^2} + (-1)^i\right) J_m^2(y) \right], \quad i = 1, 2, \tag{11.7}$$

and $p_3 = \varepsilon - p_1 - p_2$. The latter corresponds to the zero trace of the vacuum EMT. The quantities (11.5) are divergent. To make them finite we introduce the cutoff function $\psi_\mu(\gamma)$ and consider the finite quantities

$$q_c = \frac{1}{8\pi^2 a^4} \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{+\infty} dk \sum_{\lambda=0}^{1} \sum_{n=1}^{\infty} \frac{j_{m,n}^{(\lambda)3} \psi_\mu(j_{m,n}^{(\lambda)}/a)}{\sqrt{k^2a^2 + j_{m,n}^{(\lambda)2}}} T_m(j_{m,n}^{(\lambda)}) f_m^{(q)}(j_{m,n}^{(\lambda)} x), \quad q = \varepsilon, p_1, p_2 \quad (11.8)$$

with $x = r/a$, $\gamma a = j_{m,n}^{(\lambda)}$ are the roots of the equations (11.2) for $\lambda = 0$ and $\lambda = 1$, correspondingly. To calculate the sums over zeros of the functions (11.2) here we use the summation formula obtained in section 2, namely the formula (3.21). Let us choose as a function $f(z)$ in GAPF

$$f(z) = \frac{z^3}{\sqrt{z^2 + k^2a^2}} \psi_\mu(z/a) f_m^{(q)}(zx). \quad (11.9)$$

This function has branch point on the imaginary axis and we have to use the version (3.21) with lower sign. Here we will assume a class of cutoff functions for which (11.9) satisfies conditions (3.4) and (3.11) uniformly with respect to μ. By using the asymptotic formulae for Bessel functions these conditions can be easily translated in terms of ψ_μ. We choose $A = 0$, $B = 1$ in the case $\lambda = 0$ and $A = 1$, $B = 0$ in the case $\lambda = 1$ (see (3.1)), and $\nu = m$. Using the relation

$$f_m^{(q)}(ye^{-\pi i/2}) = e^{2m\pi i} f_m^{(q)}(ye^{\pi i/2}) \quad (11.10)$$

we see that the subintegrand of the first integral on rhs of the formula (3.21) is proportianal to $\psi_\mu(iz/a) - \psi_\mu(-iz/a)$. Consequently after removing the cutoff ($\psi_\mu \to 1$) the contribution of the first integral will be zero. For this reason we shall write only the second integral on the right of (3.21). For the simplicity we will assume also that the cutoff function has no poles in the right-half plane. In this case the residue terms are zero. It can be seen that the residue term on the right vanishes as well. Hence by applying GAPF to the sums over zeros of Bessel functions in (11.8) and omitting the term which will vanish after the cutoff removing we obtain

$$\begin{aligned} q_c &= \frac{1}{8\pi^2} \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{+\infty} dk \left\{ \int_0^\infty \frac{z^3 \psi_\mu(z)}{\sqrt{z^2 + k^2}} f_m^{(q)}(zr) dz + \right. \\ &+ \left. \frac{e^{-m\pi i}}{\pi a^3} \int_{|ak|}^\infty \left[\frac{K_m(z)}{I_m(z)} + \frac{K_m'(z)}{I_m'(z)} \right] f_m^{(q)}(zxe^{\pi i/2}) \frac{z^3 \chi_\mu(z/a) dz}{\sqrt{z^2 - a^2k^2}} \right\}, \end{aligned} \quad (11.11)$$

where the function $\chi_\mu(y)$ is defined in (8.18). The second integral on the right of this formula vanishes in the limit $a \to \infty$, whereas the first one does not depend on a. It follows from here that the latter corresponds to the Minkowskian part without boundaries. This can be seen also directly by explicit summation over m using the formula $\sum_{m=-\infty}^{+\infty} J_{n\pm m}^2(z) = 1$. For instance, in the case of the energy density one has

$$\begin{aligned} \varepsilon^{(0)} &= \frac{1}{8\pi^2} \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} dz \frac{z^3 \psi_\mu(z)}{\sqrt{z^2 + k^2}} f_m^{(\varepsilon)}(zr) = \\ &= \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} dk \int_0^{+\infty} dz z \sqrt{z^2 + k^2} \psi_\mu(z) = \frac{1}{2\pi} \int_0^{+\infty} \omega^3 \tilde{\psi}_\mu(\omega) d\omega, \end{aligned} \quad (11.12)$$

with $\omega^2 = z^2 + k^2$. Hence GAPF allows us to extract the contribution of unbounded space without specifying the cutoff function. The remained part is finite for $x < 1$ and $\mu \to 0$, and can be written in the form

$$q_c = \frac{1}{2\pi^3 a^4} \sum_{m=0}^{\infty}{}' e^{-m\pi i} \int_0^\infty dt \int_0^\infty dy z^2 \left[\frac{K_m(z)}{I_m(z)} + \frac{K_m'(z)}{I_m'(z)} \right] \chi_\mu(z/a) f_m^{(q)}(zxe^{\pi i/2}), \quad (11.13)$$

with $z^2 = t^2 + y^2$ and $t = ka$. Here we have introduced a new integration variable y and the prime on the summation sign indicates that the $m = 0$ term is to be halved. For $q = \varepsilon$ from (11.6) one has

$$e^{-m\pi i} z^2 f_m^{(q)}(zxe^{\pi i/2}) = z^2 I_m^2(zx) + (t^2 - y^2)\left[I_m'^2(zx) + \frac{m^2}{z^2x^2} I_m^2(zx)\right] \tag{11.14}$$

and it can be easily seen that the contribution of the summand containing $t^2 - y^2$ in (11.13) is zero. Introducing the polar coordinates (z, θ) on the plane (t, y) for the EMT components inside a perfectly conducting cylindrical surface from (11.13) one finds [17]

$$q_c(a, r) = \frac{1}{4\pi^2 a^4} \sum_{m=0}^{\infty}{}' \int_0^\infty dz \chi_\mu(z/a) F_m^{(q)}(z, x), \quad r < a, \quad q = \varepsilon, p_i, \tag{11.15}$$

where the following notations are introduced

$$F_{cm}^{(q)}(z, x) = z^3 \left[\frac{K_m(z)}{I_m(z)} + \frac{K_m'(z)}{I_m'(z)}\right] \begin{cases} I_m^2(zx), & q = \varepsilon \\ (1 + m^2/z^2x^2)\, I_m^2(zx) - I_m'^2(zx), & q = p_1 \end{cases} \tag{11.16}$$

$$F_{cm}^{(p_3)}(z, x) = -F_{cm}^{(\varepsilon)}, \quad F_{cm}^{(p_2)} = 2F_{cm}^{(\varepsilon)} - F_{cm}^{(p_1)}. \tag{11.17}$$

In particular we see that inside the cylinder $\varepsilon = -p_3$. This relation is the same as in the case of the Minkowski vacuum. This is natural, as we have no constraint on z direction. On cylinder axis $(x = 0)$ the $m = 0$ term contributes only and we have

$$\varepsilon(0) = p_1(0) = p_2(0) = \frac{1}{8\pi^2 a^4} \int_0^\infty dz z^3 \left[\frac{K_0(z)}{I_0(z)} + \frac{K_0'(z)}{I_0'(z)}\right] = -0.0168 a^{-4} \tag{11.18}$$

with $q'(0) = 0$. The vacuum EMT satisfy continuity equation which can be written now as

$$\frac{dp_1}{dr} + \frac{2}{r}(p_1 - \varepsilon) = 0, \tag{11.19}$$

or in the integral form

$$E_c(r) = 2\pi \int_0^r \varepsilon(t) t dt = \pi r^2 p_1(r), \quad r < a. \tag{11.20}$$

To determine the integration constant here we have used the relations (11.18) between the EMT components on the cylinder axis. As we see the total energy per unit length inside the cylinder with radius r is equal to the radial pressure on the surface of this cylinder multiplied by the corresponding volume.

Let us consider the behavior of the functions (11.17) in two limiting cases:

1) For fixed m and large zx from the asymptotic expansions of Bessel functions we find

$$F_{cm}^{(\varepsilon)} \sim \frac{1}{2} F_{cm}^{(p_2)} \sim -\frac{z}{2x} e^{-2z(1-x)}, \quad F_{cm}^{(p_1)} \sim -\frac{1}{2x^2} e^{-2z(1-x)}. \tag{11.21}$$

2) For large m by using the uniform asymptotic expansions of Bessel functions one obtains

$$F_{cm}^{(q)}(mz, x) \sim \Phi_{cm}^{(q)}(z, x) \exp\left\{-2m[\eta(z) - \eta(zx)]\right\}, \tag{11.22}$$

with

$$\Phi_{cm}^{(\varepsilon)}(z,x) = -\frac{m}{2}z^5 t(zx)t^3(z)\left\{1-\frac{1}{12m}\left[t(z)(t^2(z)-3)+t(zx)(5t^2(zx)-3)\right]\right\} \quad (11.23)$$

$$\Phi_{cm}^{(p_1)}(z,x) = -\frac{1}{2}z^5 t^2(zx)t^3(z), \quad (11.24)$$

where the standard notations are used.

It follows from here that at cylinder surface, $r \to a$, the expressions for the EMT components are divergent and near the surface the corresponding quantities are dominated by large m. Hence to obtain the asymptotic behaviour we can use the corresponding asymptotic formulae for modified Bessel functions. Then after the elementary summation over m we find the following asymptotic behaviour

$$\varepsilon \sim \frac{1}{2}p_2 \sim \frac{-1}{60\pi^2 a(a-r)^3}, \quad p_1 \sim \frac{-1}{60\pi^2 a^2(a-r)^2}. \quad (11.25)$$

This formulae are special cases of the general expansions for the EMT near a smooth boundary of arbitrary shape [45]. Note that, as it follows from (11.21) now, unlike the spherical case, in (11.15) with $\chi_\mu = 1$ the integral over z diverges for $x = 1$.

The results of the numerical calculations for vacuum EMT components (11.15) are presented in [17, 12]. Note that $\varepsilon, p_i < 0$, $i = 1,2$ everywhere inside the cylinder. The ratio of the energy density to the azimuthal pressure is a decreasing function on r and $0.5 \leq \varepsilon/p_2 \leq 1$.

12 Vacuum EMT outside a perfectly conducting cylinder

First we consider the vev of the electromagnetic EMT in the region between two coaxial cylindrical surfaces with radii a and b, $a < b$. The corresponding eigenfunctions have the form (11.1) with replacement $J_m(\gamma r) \to P_{\lambda m}(\gamma a, \gamma r)$, where

$$P_{\lambda m}(x,y) = \begin{cases} J_m(y)Y_m(x) - Y_m(y)J_m(x), & \lambda = 1 \\ J_m(y)Y'_m(x) - Y_m(y)J'_m(x), & \lambda = 0 \end{cases} \quad (12.1)$$

From the boundary conditions on $r = a, b$ one obtains that the eigennumbers γ have to be solutions to the following equations

$$P_{1m}(\gamma a, \gamma r)|_{r=b} = 0, \quad \lambda = 1 \quad (12.2)$$

$$\left[\frac{\partial}{\partial r}P_{0m}(\gamma a, \gamma r)\right]_{r=b} = 0, \quad \lambda = 0 \quad (12.3)$$

These equations have infinite number of simple real solutions. Now the normalization coefficients $\beta_{\lambda m}$ are in form

$$\beta_{\lambda m}^2 = \frac{\pi z^4}{4a^4\omega}\begin{cases} \left[J_m^2(z)/J_m^2(z\eta)-1\right]^{-1}, & \lambda = 1 \\ \left[\left(1-m^2/z^2\eta^2\right)J_m'^2(z)/J_m'^2(z\eta)-1+m^2/z^2\right]^{-1}, & \lambda = 0 \end{cases} \quad (12.4)$$

where $z = \gamma a$, $\eta = b/a$.

From Eq.(8.1) it follows that the vacuum EMT has diagonal form (11.4) with components

$$q_c(a,b,r) = \frac{1}{8\pi}\sum_{m=-\infty}^{+\infty}\int_{-\infty}^{+\infty}dk\sum_{\gamma,\lambda}\beta_{\lambda m}^2 f_{\lambda m}^{(q)}(\gamma a, \gamma r), \quad q_c = \varepsilon,\, p_i, \quad (12.5)$$

where the expressions for the functions $f_{\lambda m}^{(q)}(\gamma a, y)$ are obtained from (11.6) and (11.7) replacing $J_m(y) \to P_{\lambda m}(\gamma a, y)$.

The eigenvalue equations (12.2) and (12.3) can be written in terms of the function (4.1) as

$$C_m^{AB}(\eta, \gamma b) = 0, \quad A = \lambda,\ B = 1-\lambda, \quad \lambda = 0, 1 \tag{12.6}$$

(see the notation (3.1)). Note that the normalization coefficients can be expressed in terms of the function (4.6):

$$\beta_{\lambda m}^2 = \frac{\pi z^{5-2\lambda}}{4a^4\omega} T_m^{AB}(\eta, z). \tag{12.7}$$

Using these relations and introducing a cutoff function ψ_μ the divergent quantities (12.5) can be written in the form of the following finite integrosums

$$q_c = \frac{1}{32a^3} \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{+\infty} dk \sum_{\lambda=0}^{1} \sum_{n=1}^{\infty} \frac{(\gamma_{m,n}^{(\lambda)})^{5-2\lambda} \psi_\mu(\gamma_{m,n}^{(\lambda)}/a)}{\sqrt{k^2a^2 + \gamma_{m,n}^{(\lambda)2}}} T_m^{AB}(\eta, \gamma_{m,n}^{(\lambda)}) f_{\lambda m}^{(q)}(\gamma_{m,n}^{(\lambda)}, \gamma_{m,n}^{(\lambda)} x), \tag{12.8}$$

where $q_c = \varepsilon, p_i$ and $\gamma a = \gamma_{m,n}^{(\lambda)}$, $n = 1, 2, \ldots$ are the solutions to the eigenvalue equations (12.2), (12.3) or (12.6). By choosing in the formula (4.13)

$$h(z) = \frac{z^{5-2\lambda}}{\sqrt{z^2 + k^2a^2}} \psi_\mu(z/a) f_{\lambda m}^{(q)}(z, zx). \tag{12.9}$$

(as noted above this formula is valid in the case when the corresponding function has branch point on the imaginary axis (see also Remark to the Theorem 2)) one obtains

$$\begin{aligned} \sum_{n=1}^{\infty} h(\gamma_{m,n}^{(\lambda)}) T_m^{AB}(\eta, \gamma_{m,n}^{(\lambda)}) &= \frac{2}{\pi^2} \int_0^\infty \frac{h(x)dx}{\bar{J}_m^2(x) + \bar{Y}_m^2(x)} - \\ &- \frac{1}{2\pi} \int_0^\infty \frac{\bar{K}_m(\eta x)}{\bar{K}_m(x)} \frac{\left[h(xe^{\pi i/2}) + h(xe^{-\pi i/2})\right] dx}{\bar{K}_m(x)\bar{I}_m(\eta x) - \bar{K}_m(\eta x)\bar{I}_m(x)}. \end{aligned} \tag{12.10}$$

Here in accordance with (3.1) and (12.6)

$$\bar{J}_m(z) = J_m(z), \quad \lambda = 1 \tag{12.11}$$

$$\bar{J}_m(z) = zJ'_m(z), \quad \lambda = 0, \tag{12.12}$$

and in similar way for other Bessel functions in (12.10). To obtain the EMT components for the outside region of a perfectly conducting cylindrical shell we consider the limit $b \to \infty$. It can be seen that the second sum on the right of (12.10) is zero in this limit and the first one does not depend on b. Hence for the outside region of a single cylinder we obtain

$$q_c(a, r) = \frac{1}{16\pi^2 a^4} \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{+\infty} dk \int_0^\infty dz \sum_{\lambda=0,1} \frac{z^3 \psi_\mu(z/a)}{\sqrt{k^2 + z^2/a^2}} \frac{f_{\lambda m}^{(q)}(z, zx)}{\bar{J}_m^2 + \bar{Y}_m^2}. \tag{12.13}$$

To regularize we subtract from these quantities the contribution of unbounded Minkowski spacetime which can be presented in the form (see (11.12)):

$$q^{(0)} = \frac{1}{8\pi^2 a^4} \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{+\infty} dk \int_0^\infty dz \frac{z^3 \psi_\mu(z/a)}{\sqrt{k^2 + z^2/a^2}} f_m^{(q)}(zx) \tag{12.14}$$

with the function $f_m^{(q)}$ defined as (11.6), (11.7). By using the definitions of $f_{\lambda m}^{(q)}$ and $f_m^{(q)}$ it is easy to see that

$$\frac{f_{\lambda m}^{(q)}(z, zx)}{\bar{J}_m^2 + \bar{Y}_m^2} - f_m^{(q)}(zx) = -\frac{1}{2} \sum_{n=1,2} \Omega_{\lambda m}^{(n)}(z) f_m^{(nq)}(zx), \tag{12.15}$$

where by definition the expression for $f_m^{(nq)}(zx)$ is obtained from that for $f_m^{(q)}(zx)$ replacing $J_m(zx) \to H_m^{(n)}(zx)$, $n = 1, 2$, and

$$\Omega_{\lambda m}^{(n)}(z) = \begin{cases} J_m(z)/H_m^{(n)}(z), & \lambda = 1 \\ J_m'(z)/H_m^{(n)\prime}(z), & \lambda = 0 \end{cases} \tag{12.16}$$

Hence

$$\operatorname{reg} q_c(a,r) = -\frac{1}{16\pi^2 a^4} \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{+\infty} dk \int_0^{\infty} dz \sum_{\lambda,n} \Omega_{\lambda m}^{(n)}(z) f_m^{(nq)}(zx). \tag{12.17}$$

By rotating the integration contour for z by angle $\pi/2$ for $n = 1$ and by angle $-\pi/2$ for $n = 2$ (note that the function $H_m^{(1)}(z)$ ($H_m^{(2)}(z)$) has no zeros for $0 \leq \arg z \leq \pi/2$ ($-\pi/2 \leq \arg z \leq 0$)) and introducing Bessel modified functions for the regularized components we obtain (the reg sign is suppressed)

$$\begin{aligned} q_c &= \frac{1}{16\pi^3 a^4} \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{+\infty} dk \left\{ i \int_0^{|ak|} dz \left[\psi_\mu\left(\frac{iz}{a}\right) - \psi_\mu\left(-\frac{iz}{a}\right) \right] \frac{F_{cm}^{(q)}(z,x)}{\sqrt{k^2 - z^2/a^2}} + \right. \\ &\left. +2 \int_{|ak|}^{\infty} dz \chi_\mu\left(\frac{z}{a}\right) \frac{F_{cm}^{(q)}(z,x)}{\sqrt{z^2/a^2 - k^2}} \right\} \end{aligned} \tag{12.18}$$

(the definition of the functions $F_{cm}^{(q)}$ see below). In (12.18) the integrals are convergent for $x > 1$ and $\mu = 0$ and hence the cutoff can be removed. In this limit the first integral is zero and for regularized components of EMT after transformations, similar to the interior case, one obtains

$$q_c(a,r) = \frac{1}{4\pi^2 a^4} {\sum_{m=0}^{\infty}}' \int_0^{\infty} dz \chi_\mu(z/a) F_{cm}^{(q)}(z,x), \quad r > a, \quad q = \varepsilon,\, p_i, \tag{12.19}$$

where for $x > 1$ the functions $F_{cm}^{(q)}(z,x)$ are defined as

$$F_{cm}^{(q)}(z,x) = z^3 \left[\frac{I_m(z)}{K_m(z)} + \frac{I_m'(z)}{K_m'(z)} \right] \begin{cases} K_m^2(zx), & q = \varepsilon \\ \left(1 + m^2/z^2x^2\right) K_m^2(zx) - K_m'^2(zx), & q = p_1 \end{cases} \tag{12.20}$$

$$F_{cm}^{(p_3)}(z,x) = -F_{cm}^{(\varepsilon)}, \quad F_{cm}^{(p_2)} = 2F_{cm}^{(\varepsilon)} - F_{cm}^{(p_1)}. \tag{12.21}$$

It follows from here that $\varepsilon = -p_3$.

The asymptotic expressions for the functions (12.21) are as follows:

1) For fixed m and large z:

$$F_{cm}^{(\varepsilon)} \sim \frac{1}{2} F_{cm}^{(p_2)} \sim \frac{z}{2x} e^{2z(1-x)}, \quad F_{cm}^{(p_1)} \sim -\frac{1}{2x^2} e^{2z(1-x)}. \tag{12.22}$$

2) For large m from the uniform asymptotic expansions of Bessel functions one obtains

$$\begin{aligned} F_{cm}^{(\varepsilon)}(mz,x) &\sim \frac{m}{2} z^5 t(zx) t^3(z) \left\{ 1 + \frac{1}{12m} \left[t(z)(t^2(z) - 3) + t(zx)(5t^2(zx) - 3) \right] \right\} \times \\ &\times \exp\left\{ 2m[\eta(z) - \eta(zx)] \right\}, \end{aligned} \tag{12.23}$$

$$F_{cm}^{(p_1)}(mz,x) \sim -\frac{1}{2} z^5 t^2(zx) t^3(z) \exp\left\{ 2m[\eta(z) - \eta(zx)] \right\}, \tag{12.24}$$

As in the case of the interior components of the vacuum EMT is divergent when $r \to a$ with asymptotic behaviour

$$\varepsilon \sim \frac{1}{2} p_2 \sim \frac{1}{60\pi^2 a(r-a)^3}, \quad p_1 \sim \frac{-1}{60\pi^2 a^2 (r-a)^2} \tag{12.25}$$

Comparing with (11.25) we see that in calculating the total energy for the infinitely thin cylindrical shell the leading divergences cancel.

The asymptotic expressions for the vev at large distances from the cylinder axis, $r \gg a$, can be found from (12.19) introducing new integration variable $y = zx$ and expanding the integrands over $1/x$. In this limit the main contribution comes from the lowest order mode with $m = 0$ and one obtaines

$$q_c(a,r) \sim \frac{c^{(q)}}{8\pi^2 r^4 \ln(r/a)}, \quad c^{(\varepsilon)} = -c^{(p_1)} = \frac{1}{3}, \ c^{(p_2)} = 1, \quad r \gg a \tag{12.26}$$

Here compared to the spherical case the corresponding quantities tend to zero more slowly.

From the continuity equation for the vacuum EMT one has the following integral relation

$$p_1(r) = \frac{2}{r^2}\int_\infty^r \varepsilon(t)tdt = -\frac{E_c^{out}(r)}{\pi r^2}, \tag{12.27}$$

where $E_c^{out}(r)$ is the total energy (per unit length) outside cylinder with radius r. Combining this relation with (11.20) for the total vacuum energy of the cylindrical shell per unit length we obtain

$$E_c = E_c^{in}(a) + E_c^{out}(a) = \pi a^2 \left[p_1(a-) - p_1(a+)\right]. \tag{12.28}$$

By taking into account the corresponding expressions for the radial pressure this yields

$$\begin{aligned} E_c &= -\frac{1}{4\pi a^2}\sum_{m=0}^{\infty}{}' \int_0^\infty dz \chi_\mu(z/a)(\ln[I_m(z)K_m(z)])' \left[z^2 + \left(z^2+m^2\right)\frac{I_m(z)K_m(z)}{I'_m(z)K'_m(z)}\right] = \\ &= -\frac{1}{4\pi a^2}\sum_{m=0}^{\infty}{}' \int_0^\infty dz \chi_\mu(z/a) z^2 \frac{d}{dz}\ln\left[1 - z^2 (I_m(z)K_m(z))'^2\right]. \end{aligned} \tag{12.29}$$

In the last expression integrating by part and omitting the boundary term we obtain the Casimir energy in the form used in numerical calculations. The corresponding results are presented in [49, 52, 54]. Note that in the evaluation of the Casimir energy for a perfeclty conducting cylindrical shell by Green function method to perform the complex frequency rotation procedure an additional cutoff function have to be introduced (see [49]). This is related to the abovmentioned divergency of the integrals over z for $x = 1$.

The results of the numerical evaluations for the energy density and pressures distributions (formula (12.19)) are presented in [17, 12]. The energy density and azimuthal pressure in the exterior region are always positive, and radial pressure is negative. The ratio of the energy density to the azimuthal pressure is decreasing function on r, and $1/3 \le \varepsilon/p_2 \le 0.5$. Note that this ratio is continous function for all r and monotonically decreases from 1 at the cylinder axis to 1/3 at infinity.

13 Vacuum EMT between two coaxial cylindrical shells

By using the results from previous section the vev of the electromagnetic EMT in the region between two coaxial conducting cylindrical surfaces may be presented in the form (11.4) with components

$$q_c(a,b,r) = q_c(a,r) + q_c^{(ab)}(r), \quad a < r < b, \quad q_c = \varepsilon, \, p_i, \tag{13.1}$$

where $q_c(a,r)$ is given by (12.13), and

$$q_c^{(ab)}(r) = -\frac{1}{16\pi a^3}\sum_{m=0}^{\infty}{}' \int_0^\infty dk \sum_{\lambda=0}^{1}\int_0^\infty \frac{\bar K_m(z\eta)}{\bar K_m(z)} \frac{\left[h(ze^{\pi i/2}) + h(ze^{-\pi i/2})\right]dz}{\bar K_m(z)\bar I_m(z\eta) - \bar K_m(z\eta)\bar I_m(z)}. \tag{13.2}$$

Here the function $h(z)$ is defined according to (12.9). As we have shown the first summand on the right of (13.1) presents the corresponding quantity for the vacuum outside a single perfectly conducting cylindrical shell with radius a. As we shall see later $q^{(ab)}(r)$ is finite for $a \leq r < b$ at $\mu = 0$, and hence regularization is necessary for $q_c(a, r)$ only. This have been done in previous section. We have shown that result does not depend on specific form of the cutoff function and can be presented in the form (12.19) and (12.21).

In (13.2) the integral over z can be presented as a sum of two integrals along segments $(0, |ak|)$ and $(|ak|, \infty)$. By using the relation (3.20) and the explicit form of $h(z)$ it is easy to see that the first integral will contain the cutoff function in the form $\psi_\mu(iz/a) - \psi_\mu(-iz/a)$ and hence vanishes after the cutoff removing. For this reason below we will consider the second integral only. After the transformations similar to those we used to obtain (11.15), the quantities $q^{(ab)}$ can be written in the form

$$q_c^{(ab)}(r) = \frac{1}{4\pi^2 a^4} \sum_{m=0}^{\infty}{}' \int_0^\infty dz \sum_{\lambda=0}^{1} z^3 \Omega_{\lambda m}^c(\eta, z) F_{\lambda m}^{(q)}(z, zx), \quad x = r/a, \tag{13.3}$$

where

$$\Omega_{1m}^c(\eta, z) = \frac{K_m(z\eta)/K_m(z)}{K_m(z)I_m(z\eta) - K_m(z\eta)I_m(z)}, \tag{13.4}$$

$$\Omega_{0m}^c(\eta, z) = \frac{K_m'(z\eta)/K_m'(z)}{K_m'(z)I_m'(z\eta) - K_m'(z\eta)I_m'(z)}, \tag{13.5}$$

and

$$\begin{aligned} F_{c\lambda m}^{(\varepsilon)}(z, y) &= Q_{\lambda m}^2(z, y), \quad F_{c\lambda m}^{(p_3)} = F_{c\lambda m}^{(\varepsilon)} - F_{c\lambda m}^{(p_1)} - F_{c\lambda m}^{(p_2)} \\ F_{c\lambda m}^{(p_i)}(z, y) &= \left(1 - (-1)^i \frac{m^2}{y^2}\right) Q_{\lambda m}^2(z, y) + (-1)^i \left[\frac{\partial}{\partial y} Q_{\lambda m}(z, y)\right]^2, \quad i = 1, 2 \end{aligned} \tag{13.6}$$

Here we have introduced the notation

$$\begin{aligned} Q_{1m}(z, y) &= K_m(z)I_m(y) - I_m(z)K_m(y) \\ Q_{0m}(z, y) &= K_m'(z)I_m(y) - I_m'(z)K_m(y). \end{aligned} \tag{13.7}$$

The quantities (13.1) with (12.19) and (13.3) present the regularized vev of the EMT components in the region between two coaxial conducting cylindrical surfaces. Let us consider the limiting cases of the term (13.3). First let $a/r,\ a/b \ll 1$. After replacing $z \to z\eta$ and expanding the subintegrand over a/r and a/b it can be seen that

$$q_c^{(ab)}(r) \approx q_c(b, r), \quad a/r,\ a/b \ll 1,\ r < b, \tag{13.8}$$

where $q_c(b, r)$ are the components for the vacuum EMT inside a single cylindric shell with radius b (see (11.15), (11.16)).

When $a \to b$ the sum over m in (13.3) diverges. Consequently for $b - a \ll b$ the main contribution to $q_c^{(ab)}$ is due to large m. By using the uniform asymptotic expansions for Bessel functions in this limit one obtaines

$$\varepsilon \approx -\frac{1}{2\pi^2 a^4} \sum_{m=0}^{\infty}{}' \int_0^\infty \frac{m^3 z^3 dz}{e^{2m[\eta(zb/a) - \eta(z)]} - 1} \approx -\frac{\pi^2}{720(b-a)^4}, \tag{13.9}$$

which coincides with the corresponding quantity for the Casimir parallel plate configuration.

From (13.1) and (13.3) it can be seen that the vev of EMT components can be written also in the form

$$q_c(a, b, r) = q_c(b, r) + \tilde{q}_c^{(ab)}(r), \quad a < r < b, \quad q_c = \varepsilon, p_i. \tag{13.10}$$

Here $q_c(b,r)$ are vev inside a single cylindrical surface with radius b (see (11.15), (11.16) with replacement $a \to b$), and

$$\tilde{q}_c^{(ab)}(r) = \frac{1}{4\pi^2 b^4} \sum_{m=0}^{\infty}{}' \int_0^{\infty} dz \sum_{\lambda=0}^{1} z^3 \tilde{\Omega}_{\lambda m}^c(\sigma, z) F_{\lambda m}^{(q)}(z, zy) \tag{13.11}$$

where $y = r/b$, $\sigma = a/b$, and

$$\begin{aligned} \tilde{\Omega}_{1m}^c(\sigma, z) &= \frac{I_m(z\sigma)/I_m(z)}{I_m(z)K_m(z\sigma) - I_m(z\sigma)K_m(z)}, & (13.12) \\ \tilde{\Omega}_{0m}^c(\sigma, z) &= \frac{I_m'(z\sigma)/I_m'(z)}{I_m'(z)K_m'(z\sigma) - I_m'(z\sigma)K_m'(z)}. & (13.13) \end{aligned}$$

The quantites (13.11)are finite for all $a < r \leq b$ and diverge on surface $r = a$.

From the above it follows that if we present the vacuum EMT components between cylindrical surfaces in the form

$$q_c = q_c(a,r) + q_c(b,r) + \Delta q_c(a,b,r) \tag{13.14}$$

then the quantities

$$\Delta q_c(a,b,r) = q_c^{(ab)}(r) - q_c(b,r) = \tilde{q}_c^{(ab)}(r) - q_c(a,r) \tag{13.15}$$

are finite for all r from $a \leq r \leq b$. In (13.15) the first presentation is convenient near the surface $r = a$, as in this case both summands are finite. Similarly the second presentation is convenient near $r = b$.

Let us consider a system of two coaxial thin cylindrical shells with radii a and b, $a < b$. The vacuum EMT components may be written in the form

$$q_c = q_c(a,r)\theta(a-r) + q_c(b,r)\theta(r-b) + \left[q_c(a,r) + q_c^{ab}(r)\right]\theta(r-a)\theta(b-r). \tag{13.16}$$

Similar to the spherical case using the continuity equation (11.19) the total Casimir energy for this system may be written as

$$E_c^{(ab)} = E_c(a) + E_c(b) + \pi b^2 \tilde{p}_{c1}^{(ab)}(b) - \pi a^2 p_{c1}^{(ab)}(a), \tag{13.17}$$

where $E_c(i)$ is the Casimir energy for a single cylindrical shell with radius i, $i = a, b$. For the additional vacuum pressures on the cylindrical surfaces from (13.3) and (13.11) one has:

$$\begin{aligned} p_{c1}^{(ab)}(a) &= \frac{1}{4\pi^2 a^4} \sum_{m=0}^{\infty}{}' \int_0^{\infty} z dz \left[\left(\frac{m^2}{z^2} + 1\right) \Omega_{0m}^c(\eta, z) - \Omega_{1m}^c(\eta, z)\right], & (13.18) \\ \tilde{p}_{c1}^{(ab)}(b) &= \frac{1}{4\pi^2 b^4} \sum_{m=0}^{\infty}{}' \int_0^{\infty} z dz \left[\left(\frac{m^2}{z^2} + 1\right) \tilde{\Omega}_{0m}^c(\sigma, z) - \tilde{\Omega}_{1m}^c(\sigma, z)\right], & (13.19) \end{aligned}$$

where $\Omega_{\lambda m}^c(\eta, z)$ and $\tilde{\Omega}_{1m}^c(\sigma, z)$ are defined in (13.4), (13.5), (13.12) and (13.13). In (13.18) and (13.19) the first summands in braces come from the magnetic waves contribution, and second ones from the electric type waves.

Let us now consider the interaction forces between cylindrical surfaces. The force acting per unit area of the inner surface can be presented in the form

$$F_c^{(a)} = F_{c1}^{(a)} + \Delta F_c^{(a)}, \quad \Delta F_c^{(a)} = -p_c^{(ab)}(a), \tag{13.20}$$

where $F_{c1}^{(a)}$ is the force acting on a single cylindrical surface with radius a, and $\Delta F_c^{(a)}$ is additional force due to the existence of the outer surface and is determined from (13.18). The latter is finite without additional subtractions.

By similar way the force acting per unit area of the outer cylinder

$$F_c^{(b)} = F_{c1}^{(b)} + \Delta F_c^{(b)}, \quad \Delta F_c^{(b)} = \tilde{p}_c^{(ab)}(b) \tag{13.21}$$

where additional term $\Delta F_c^{(b)}$ is due to the existence of the inner cylinder and is defined by (13.19).

The results of the numerical calculations for quantities $\Delta q(a, b, r)$ are given in [18]. Note that the sign of $\Delta\varepsilon$ and Δp_{c1} is the same as in the case of interior of the parallel plate configuration. In particular the additional forces $\Delta F_c^{(a,b)}$ always have attractive nature.

14 Summary

In the present paper we considered a possible way for generalization of Abel-Plana summation formula, proposed in [11]. The generalized version contains two meromorphic functions $f(z)$ and $g(z)$ and is formulated in the form of Theorem 1. The special choice $g(z) = -if(z)\cot\pi z$ with $f(z)$ being an analytic function in the right half-plane gives APF with additional residue terms coming from the poles of $f(z)$. Another consequence from GAPF is the summation formula (2.17) over the points with integer values of an analytic function. An application of this formula to the Casimir effect is given in [10].

Further we consider the applications to the series and integrals involving Bessel functions. First of all, in section 3 choosing the function $g(z)$ in the form (3.2) we derive two types of summation formulae for the series $\sum_k T_\nu(\lambda_{\nu,k})f(\lambda_{\nu,k})$ (the definition $T_\nu(z)$ see (3.8)) with $\lambda_{\nu,k}$ being the zeros of the function $\bar{J}_\nu(z) = AJ_\nu(z) + BzJ'_\nu(z)$. Such a type of series arises in a number of problems of mathematical physics with spherical and cylindrical symmetry. As a special case they include Fourier-Bessel and Dini series (see [20]). Using the formula (3.18) the difference between the sum over zeros of $\bar{J}_\nu(z)$ and corresponding integral can be presented in terms of an integral involving Bessel modified functions plus residue terms. For a large class of functions the last integral converges exponentially fast and is useful for numerical calculations. The mode summation method for calculating the vev of the EMT inside perfectly conducting spherical and cylindrical shells used in [13, 14, 17] is based on this formula. In this method the independence of the regularized EMT components on specific form of the cutoff function becomes obvious. APF is a special case of (3.18) with $\nu = 1/2$, $A = 1$, $B = 0$ and an analytic function $f(z)$. Choosing $\nu = 1/2$, $A = 1$, $B = 2$ we obtain APF in the form (2.15) useful for fermionic field calculations. Note that the formula (3.18) may be used also for some functions having poles and branch points on the imagianary axis.

The second type of summation formulae, formula (3.33), considered in subsection 3.2 (Theorem 3), is valid for functions satisfying condition (3.27) and presents the difference between the sum over zeros of $\bar{J}_\nu(z)$ and corresponding integral in terms of residues over poles for $f(z)$ in the right half-plane (including purely imaginary ones). It may be used to summarize a large class of series of this type in finite terms. In particular, the examples we found in literature, when the corresponding sum may be presented in closed form, are special cases of this formula. A number of new series summable by this formula and some classes of functions to which it can be applied is presented.

In Section 4 we consider applications to the series of type $\sum_k T_\nu^{AB}(\lambda, \gamma_{\nu,k})h(\gamma_{\nu,k})$ (with $T_\nu^{AB}(\lambda, z)$ defined as (4.6)), where $\gamma_{\nu,k}$ are zeros of the function $\bar{J}_\nu(z)\bar{Y}_\nu(\lambda z) - \bar{J}_\nu(\lambda z)\bar{Y}_\nu(z)$. The corresponding results are formulated in the form of Corollary 2 and Corollary 3. Using the formula (4.13) the difference between the sum and corresponding integral can be expressed as an integral containing Bessel modified functions plus residue terms. For the large class of functions $h(z)$ this integral converges exponentially fast. The formula of the second type, (4.16), allows to find in closed form the sums of some types of the series over $\gamma_{\nu,k}$. To evaluate the

corresponding integral the formula can be used derived in section 7. This yields to the another summation formula, (4.19), containing residue terms only. The similar formulae can be obtained for the series over zeros of the function $J'_\nu(z)Y_\nu(\lambda z) - J_\nu(\lambda z)Y'_\nu(z)$ as well. Note that the several examples we found in literature when the corresponding sum was evaluated in closed form are special cases of the formulae considered here. We present new examples and some classes of functions satisfying the corresponding conditions. The possibilities are endless.

The results from GAPF for the integrals of type p.v. $\int_0^\infty F(x)\bar{J}_\nu(x)dx$ (see notation (3.1)) and p.v. $\int_0^\infty F(x)[J_\nu(x)\cos\delta + Y_\nu(x)\sin\delta]dx$ are considered in section 5. The corresponding formulae have the form (5.4), (5.7) and (5.18). In particaular the formula (5.4) is useful to express the integrals containing Bessel functions with oscillating subintegrand through the integrals of modified Bessel functions with exponentially fast convergence. The results obtained in [24] are special cases of these formulae. The illustrating examples of applications of the formulae for integrals are given in section 6 (see (6.3)-(6.7) and (6.13)- (6.17)). Looking the standard books (see, e.g., [7], [20]-[29]) one will find many particular cases which follow from these formulae. Many new integrals can be evaluated as well. We consider also two examples of functions having purely imaginary poles, (6.19) and (6.23), with corresponding formulae (6.20) and (6.24) (two special cases of these formulae see [20]).

By choice of the functions $f(z)$ and $g(z)$ in accord with (7.1) formulae (7.7) and (7.16) for integrals of type

$$\text{p.v.} \int_0^\infty \frac{J_\nu(x)Y_\mu(\lambda x) - J_\mu(\lambda x)Y_\nu(x)}{J_\nu^2(x) + Y_\nu^2(x)} F(x)dx$$

can be derived from GAPF. The corresponding results are formulated in the form of Theorem 5 and Theorem 6 in section 7. The several examples for the integrals of this type we have been able to find in literature are particular cases of the formula (7.7). New examples when the integral is evaluated in finite terms are presented. Some classes of functions are distinguished to which the corresponding formulae may be applied.

In the following sections, based on [13]-[18], the physical applications of the summation formulae obtained from GAPF are reviewed. We consider the vacuum expectation values of the energy-momentum tensor for the electromagnetic field inside and outside the perfectly conducting spherical and cylindrical shells, as well as between two conducting cocentric spherical and coaxial cylindrical surfaces.The corresponding mode sums contain the series over zeros of Bessel functions and their combinations. The application of the summation formulae from sections 3 and 4 allows (i) to extract from corresponding divergent quantities the contribution of the unbounded space in explicitly cutoff independent way, and (ii) to obtain for the regularized values strongly convergent integrals. To compare note that in the Green function method after the subtraction of the Minkowskian part the additional complex frequency rotation is used. In the regularization scheme based on the summation formulae of APF type the complex frequency rotation is made automatically. The corresponding global quantities such as total Casimir energy or forces acting on the surfaces can be obtained from the EMT components. It is shown that in the geometries with two surfaces the additional vacuum forces due to the existence of the second surface always have attractive nature. In the limiting case of the large radii the corresponding results for the Casimir parallel plate configuration are obtained.

Of course the applications of the summation formulae obtained from GAPF are not restricted by the Casimir effect only. Similar types of series will arise in considerations of various physical phenomenon near the boundaries with spherical and cylindrical symmetries, for example in calculations of the electron self-energy and the electron anomalous magnetic moment (for the similar problems in the plane boundary case see, e.g., [56, 57] and references therein). The dependence of these quantities on boundaries originates from the modification of the photon propagator due to the boundary conditions imposed by the walls of the cavity.

Acknowledgements

I am indebted to Prof. G. Sahakyan, Prof. E. Chubaryan and Prof. A. Mkrtchyan for general encouragement, valuable comments and suggestions. I am grateful to Levon Grigoryan for fruitful colaboration and stimulating discussions. A part of this paper was written during my stay in Tehran, when I given lectures on the Casimir effect at Sharif University of Technology. I acknowledge the Physics Department and Prof. Reza Mansouri for the hospiality. I would also like acknowledge the hospitality of the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

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